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| **E. Kreyszig, Advanced Engineering Mathematics, $9^{th}$ ed.** | Section 8.3, pgs. 345-349 |

Lecture: Special Matrices Module: 08

Suggested Problem Set: Suggested Problems : {5, 7, 10, 17} Last Compiled: February 16, 2010

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| **E. Kreyszig, Advanced Engineering Mathematics, $9^{th}$ ed.** | Section 8.4, pgs. 349-356 |

Lecture: Diagonalization Module: 08

Suggested Problem Set: Suggested Problems : {4, 7, 12, 15, 17} Last Compiled: February 16, 2010

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| **E. Kreyszig, Advanced Engineering Mathematics, $9^{th}$ ed.** | Section 8.5, pgs. 356-363 |

Lecture: Complex Matrices Module: 08

Suggested Problem Set: Suggested Problems : {5, 6, 9, 12, 16} Last Compiled: February 16, 2010

| Quote of Lecture 8 | |
|---|---|
| When I get to the bottom I go back to the top of the slide. Where I stop and turn and I go for a ride 'till I get to the bottom and I see you again. | |
| | The Beatles : The White Album (1968) |

At this point we have the system $\mathbf{Ax} = \lambda\mathbf{x}$, which is an eigenvalue/eigenvector problem for the square matrix $\mathbf{A}$. We solve this problem by finding roots of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and then defining a basis for the null-space of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ we find the eigenvectors of $\mathbf{A}$. Now, we consider what can be done with this information but before this we present two statements important to this development:

1. We say that two matrices are **similar** if there exists $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PBP}^{-1}$ and we call the previous equality a **similarity transformation** of the matrix $\mathbf{A}$. This implies two things:
   (a) $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{B} - \lambda\mathbf{I})$[1]
   (b) $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$
2. Diagonalization Theorem: Let $\mathbf{A}$ be a square-matrix such that the algebraic multiplicity is equal to the geometric multiplicity for each eigenvalue.[2] Then there exists an invertible matrix $\mathbf{P}$ and diagonal matrix $\mathbf{D}$ such that,
   $$\mathbf{A} = \mathbf{PDP}^{-1},$$
   where $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{A}$ and $\mathbf{D}$ is a diagonal matrix whose elements are the eigenvalues of $\mathbf{A}$ corresponding to the 'eigen-columns' of $\mathbf{P}$. [3] Notes:
   - We call this particular similarity transformation the **diagonalization** of $\mathbf{A}$.
   - The conclusion is that if we have enough eigenvectors then the eigenbasis can be used as a coordinate system for the column space of the matrix and under this coordinate system the matrix is diagonal and is populated with its own spectrum.

[1] To see this we consider the determinant of $\mathbf{B} - \lambda\mathbf{I}$ to find $\det(\mathbf{B} - \lambda\mathbf{I}) = \det(\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}) = \det(\mathbf{A} - \lambda\mathbf{I})$.

[2] This implies that the $\mathbf{A}_{n\times n}$ gives rise to $n-$many eigenvectors. If this were not the case then we wouldn't know how to populate the $\mathbf{P}$ matrix.

[3] It can be shown that eigenvectors are naturally linearly independent, which implies that the columns of $\mathbf{P}$ form a linearly independent set and thus $\mathbf{P}$ is an invertible matrix.

- If it is also true that if $\mathbf{A}$ is self-adjoint, $\mathbf{A} = \mathbf{A}^{\text{H}}$, then the matrix $\mathbf{P}$ can be written as an orthogonal matrix $\mathbf{Q}$ and thus $\mathbf{Q}^{-1} = \mathbf{Q}^{\text{H}}$. [4]
- Using diagonalization one can then show $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ where the elements of $\mathbf{D}^k$ are given by $d_{ii}^k$. This can be used to efficiently raise a matrix to an unreasonably large power. [5]

**Lecture Goals**

- Understand when and how the eigenvalue/eigenvector data can be used to construct a diagonal decomposition of a square matrix.
- Notice how self-adjointness of a matrix can be used to make the diagonalization process more efficient.

**Lecture Objectives**

- State the hypotheses of diagonalization theorem and apply its results to previous eigen-problems for the purposes of defining similarity transformations.
- Using previous problems illustrate how certain matrices can be orthogonally diagonalized.
- Apply the diagonalization procedure to raise matrices to large powers.

[4] If we assume that the algebraic multiplicity of each eigenvalue is 1 then to find $\mathbf{Q}$ we need only normalize the eigenvectors of $\mathbf{A}$. If this is not the case then orthonormality can be forced by a process known as Gram-Schmidt, which takes linearly independent vectors and creates orthogonal vectors spanning the same space. We will not discuss this here.

[5] At this point one can even consider $\lim_{k\to\infty} \mathbf{A}^k$.