MATH 235 - Differential Equations w/ Honors March 31, 2008
Homework 6, Spring 2008 **Due**: April 11, 2008

Second Order Linear ODEs - Oscillators
Exponential Forms - Power Series - Integral Transformations

1. Consider the following second-order linear ordinary differential equation with constant coefficients,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \quad a, b, c \in \mathbb{R}. \tag{1}$$

Solve (1) for the following cases, when possible solve for any unknown coefficients,

(a) $a = 1$, $b = -2$, $c = -3$, $f(t) = 3e^{-t}$.
(b) $a = 1$, $b = 4$, $c = 4$, $f(t) = 3e^{-t} + t^2$.
(c) $a = 1$, $b = -4$, $c = -13$, $f(t) = 0$, subject to, $y(0) = 1$ and $y'(0) = -1$.
(d) $a = 1$, $b = 0$, $c = 9$, $f(t) = 2\sin(2t)$.
(e) $a = 1$, $b = 0$, $c = 9$, $f(t) = \cos(3t)$.

2. Consider the ordinary differential equation:

$$y'' - y = 0 \tag{2}$$

We know that the general solution to this equation is $y(t) = c_1e^t + c_2e^{-t}$. It is common to write the solutions to (2) in terms of the hyperbolic trigonometric functions, $\sinh(t) = \dfrac{e^t - e^{-t}}{2}$, $\cosh(x) = \dfrac{e^t + e^{-t}}{2}$.

(a) Show that $y(x) = b_1\sinh(t) + b_2\cosh(t)$ is a solution to the differential equation (2).
(b) Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(t) = c_1e^t + c_2e^{-t} = b_1\cosh(t) + b_2\sinh(t)$.
(c) Assume that $y(t) = \displaystyle\sum_{n=0}^{\infty} a_nt^n$ and find the general solution of (2) in terms of the hyperbolic sine and cosine functions. [1]

3. Recall the differential equation given by (1). For this equation we can define the Green's function for the differential equation as the function g, which satisfies the analogous differential equation, [2]

$$a\frac{d^2g}{dt^2} + b\frac{dg}{dt} + g = \delta_0(t), \quad g(0) = 0, \ g'(0) = 0. \tag{4}$$

Determine the Green's functions to (1) for:[3]

(a) $a = 1$, $b = -2$, $c = -3$
(b) $a = 1$, $b = 4$, $c = 4$
(c) $a = 1$, $b = -4$, $c = -13$
(d) $a = 1$, $b = 0$, $c = 9$

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[1] The hyperbolic sine and cosine have the following Taylor's series representations centered about $t = 0$:

$$\cosh(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \qquad \sinh(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \tag{3}$$

[2] A Green's function is a certain type of function used to solve nonhomogenous equations by considering the response of the system to a primitive external impulse, i.e. the Dirac-Delta *function*, and then using the primitive response to construct solutions for more complicated external forces. In physics Green's functions are often called propagators. In statistics a Green's function are often seen as correlation functions used to describe relationships between random variables.

[3] To do this take the Laplace transform of (4), solve for $G(s)$ and from $G(s)$ use tables to determine $g(t)$.

4. Read the following websites,
   - `http://en.wikipedia.org/wiki/Laplace_transform` - Read the introductory paragraph and skim the examples.
   - `http://en.wikipedia.org/wiki/Convolution` - Read the introductory paragraph, definition and applications.
   - `http://mathworld.wolfram.com/Convolution.html` - Read through the paragraph after the animation.

   and answer the following:

   (a) Using the Laplace transform as an example, what is an integral transform and what are two reasons for using integral transforms?

   (b) List two examples of the application of Laplace transforms to linear dynamical systems. What is the benefit of using the Laplace transforms for each case?

   (c) What is the relationship between the output of a linear dynamic system and its forcing function?

   (d) Considering the animation found on the mathworld site, when is the convolution of two Gaussian functions at a maximum?

5. Consider the governing equation for a mass suspended from an ideal spring. Including forces due to friction, and an external applied force, $f(t)$, leads to the second order linear ordinary differential equations with constant coefficients:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}, \tag{5}$$

   (a) If $b = 0$ then the oscillator is called *simple*. Show that from the homogeneous (not forced) simple harmonic oscillator one can derive the conservation law $E_{total} = \frac{mv^2}{2} + \frac{ky^2}{2}$ where $v = \frac{dy}{dt}$ and E_{total} is a constant.[4]

   (b) Assume that $m = k = 2$ and graph the conservation law in the yv-plane for $E_{total} = 1, 4, 9$. [5] [6]

   (c) Show that, for an unforced simple harmonic oscillator, the that the solution can be written as $y_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$. Determine w_0 in terms of m and k.

   (d) Let $f(t) = \cos(\alpha t)$, $\alpha \in \mathbb{R}$. Pick the form of the particular solution, $y_p(t)$, for the simple harmonic oscillator. What happens when $\alpha = w_0$? Write down the functional form of the general solution for both of these cases. (Do not solve for the undetermined coefficients)

   (e) Consider the program BeatsAndResonance where $a = 1.5$.

      i. Describe what happens to the general solution (green) as the circular frequency, ω, of forcing is changed from 0.5 through 1.5. [7]

      ii. Describe the changes to the homogenous solution (blurple) and nonhomogenous solution (red), relative to one another, as the frequency of forcing is changed from 0.5 through 1.5.

      iii. If the energy of a single cycle of a sinusoidal-wave is proportional to the square of the amplitude then compare the amount of energy in one beat envelope for when $\omega \approx 0.5$ to when $\omega \approx 1.2$. What happens to the energy when $\omega \approx 1.5$?

[4] In physics one would call this conservation law a *constant of motion*.

[5] These *constants of motion* are nothing more than trajectories of the simple harmonic oscillator in the phase-plane.

[6] Recall that we derived the model equation for a mass-spring system appealing to the classical Newtonian physics. Considering microscopic objects requires us to appeal to Schrodinger's equation, the quantum analogue of Newton's second law. In this framework we find the same conservation law. However, since Schrodinger's equation, under the harmonic oscillator potential, gives rise to Hermite's equation we have the extra consequence that the energy E_{total} can only come in particular discrete/quantized values.

[7] You may find it useful to toggle the *Envelope* feature.