HW 6

Note Title 11/18/2006

Snieder 13.4

a) orthog. of E-vectors

$$A\,V^n = \lambda_n V^n$$

$$A\,V^m = \lambda_m V^m$$

} by def.

$$\underbrace{(V^m)^T A}_{(A^T V^m)^T} V^n = \lambda_n (V^m)^T V^n$$

$$= (A\,V^m)^T$$

$$= (\lambda_m V^m)^T$$

$$= \lambda_m (V^m)^T$$

$$\lambda_m V^{mT} V^n = \lambda_n V^{mT} V^n$$

$$= (\lambda_m - \lambda_n) V^{mT} V^n = 0$$

so if $\lambda_m \neq \lambda_n$ then the E-vectors are orthog.

If $m = n$ then $v^{mT} v^m$

is just $\|v^m\|^2 = 1$

So $v^{mT} v^n = \delta_{mn}$ ✓

Reality of E-values

$$A v = \lambda v$$

take c.c.

$$\underbrace{A}_{\text{real by def}} v^* = \lambda^* v^*$$

$$\underbrace{v^T A}_{\substack{(Av)^T \\ = (\lambda v)^T \\ = \lambda v^T}} v^* = \lambda^* v^T v^*$$

$$\lambda v^T v^* = \lambda^* v^T v^*$$

$$\Rightarrow \quad \lambda = \lambda^*$$

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b) $$V^T V$$

$$= \begin{pmatrix} \cdots\cdots \hat{v}^{(1)} \cdots\cdots \\ \hat{v}^{(2)} \\ \vdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots \\ \hat{v}^1 & \hat{v}^2 \\ \vdots & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \hat{v}^{(1)T} \hat{v}^{(1)} & \hat{v}^{(1)} \hat{v}^{(2)} & \cdots \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & & \end{pmatrix} = I$$

c) $$I = \sum_{n=1}^{N} \hat{v}^{(n)} \hat{v}^{(n)T}$$

$$\Rightarrow \quad P = I P$$

$$= \left(\sum \hat{v}^{(n)} \hat{v}^{(n)T} \right) P$$

$$= \sum \hat{v}^{(n)} \hat{v}^{(n)T} p$$

$$Ap = \sum_n \underbrace{A \hat{v}^{(n)}}_{\lambda_n \hat{v}^{(n)}} \hat{v}^{(n)T} p$$

$$= \sum \lambda_n v^{(n)} \underbrace{\left(\hat{v}^{(n)T} p\right)}_{\hat{v}^{(n)} \cdot p}$$

d) $A \cdot p = \sum_n \lambda_n \hat{v}^{(n)} \hat{v}^{(n)} \cdot p$

$$\Rightarrow \left(A - \sum \lambda_n \hat{v}^{(n)} \hat{v}^{(n)T}\right) p = 0$$

if this holds for any p,

then $(A - \cdots) = 0$

$$\Rightarrow \boxed{A = \sum_n \lambda_n v^{(n)} v^{(n)T}}$$

e) $\Sigma = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$

$$V\Sigma = \begin{pmatrix} \vdots & \vdots & \\ v^1 & v^2 & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots & \vdots & \\ \lambda_1 v^1 & \lambda_2 v^2 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$V\Sigma V^T = \begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} \vdots & \vdots & \\ v^{1T} & v^{2T} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$= \lambda_1 v^1 v^{1T} + \lambda_2 v^2 v^{2T} + \cdots$$

$$= \sum \lambda_N v^{(N)} v^{(N)T}$$

13.5 $\quad A^{-1}A = I$

a)

$$\Rightarrow \quad A^{-1} A v^{(N)} = I v^{(N)} = v^{(N)}$$

$$\Rightarrow \quad A^{-1} \lambda_N v^{(N)} = v^{(N)}$$

$$\Rightarrow \quad A^{-1} v^{(n)} = \frac{1}{\lambda_n} v^{(n)}$$ ✓

b) Proof is the same as previous problem e).

c) $$A v^{(n)} = \lambda_n v^{(n)}$$

$$\Rightarrow \quad A^2 v^{(n)} = \lambda_n A v^{(n)}$$

$$= \lambda_n^2 v^{(n)}$$

$$\Rightarrow \quad A^2 v^{(n)} = \lambda_n^2 v^{(n)}$$

by induction

$$\underline{A^p v^{(n)} = \lambda_n^p v^{(n)}}$$

d) if $A = V \Sigma V^T$

then $A^2 = (V \Sigma \underbrace{V^T)(V}_{V^T V = I} \Sigma V^T)$

$$A^2 = V \Sigma \Sigma V^T$$
$$= V \Sigma^2 V^T$$

$\vdots$

by induction $\quad A^p = V \Sigma^p V^T$

e) $\quad f(z) = \sum a_p z^p$

$$f(A) = \sum a_p A^p$$
$$= \sum a_p V \Sigma^p V^T$$
$$= V \left(\sum a_p \Sigma^p \right) V^T$$

$$\begin{pmatrix} \sum a_p \lambda_1^p & & \\ & \sum a_p \lambda_2^p & \\ & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} f(\lambda_1) & & \\ & f(\lambda_2) & \\ & & \ddots \end{pmatrix}$$

def.

$$f(A) = V\, f(\Sigma)\, V^T$$

$$= \begin{pmatrix} \vdots & & \\ \hat{v}^{(1)} & \cdots & \\ & & \end{pmatrix} \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & \end{pmatrix} \begin{pmatrix} \vdots & & \\ \hat{v}^{(1)T} & & \\ & & \end{pmatrix}$$

$$= f(\lambda_1)\, \hat{v}^{(1)}\, \hat{v}^{(1)T} + f(\lambda_2) \cdots$$

$$= \sum_p f(\lambda_p)\, \hat{v}^{(p)}\, \hat{v}^{(p)T}$$

<u>13.6</u> $\mathrm{Det}\,(A - \lambda I)$

c (a,b below)

$$= \mathrm{Det} \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\big((2-\lambda)(1-\lambda) - 1\big)$$

$$+ 1(-1(1-\lambda))$$

$$\Rightarrow \quad (1-\lambda)\left[(2-\lambda)(1-\lambda) - 1 - 1\right]$$

$\lambda = 1$ is a root, else $[\ \ ] = 0$

$$2 - 3\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 3\lambda = 0$$

$$\lambda = \frac{3 \pm 3}{2} = 0, 3$$

So E-values are $\underline{0, 1, 3}$

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a) $$\ddot{x}_1 = \frac{k}{m}(x_2 - x_1) + F_1/m$$

$$\ddot{x}_2 = -\frac{k}{m}(x_2 - x_1) + \frac{k}{m}(x_3 - x_2) + \frac{F_2}{m}$$

$$\ddot{x}_3 = -\frac{k}{m}(x_3 - x_2) + F_3/m$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \frac{1}{m} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\downarrow$$

$$-w^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{k}{m} \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - w^2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{m} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\left(A - \frac{mw^2}{k} I\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

modes given by solution to
Homog. problem

$$A \vec{x} = \frac{mw^2}{k} \vec{x} = \lambda \vec{x}$$

so $\lambda = \frac{mw^2}{k}$

$\Rightarrow \quad \omega = \sqrt{\frac{k\lambda}{m}} \quad \lambda$ {- values}

of A.

d)

$A\vec{x} = \lambda \vec{x} \qquad \lambda = 0, 1, 3$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$\lambda = 0$ we have

$$A\vec{x} = 0$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + x_3 \end{bmatrix} = 0$$

$$\Rightarrow \quad \begin{aligned} x_1 &= x_2 \\ x_2 &= x_3 \end{aligned} \quad \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

normalize $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ etc.

Similarly for the other λ.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \quad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

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e) $A\vec{x} = \lambda\vec{x}$

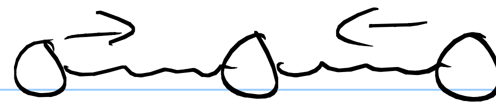
e.g. $\lambda = 0$

$$\begin{bmatrix}1 & -1 & 0\\-1 & 2 & 1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = 0\begin{bmatrix}1\\1\\1\end{bmatrix}$$

✓ same for the rest

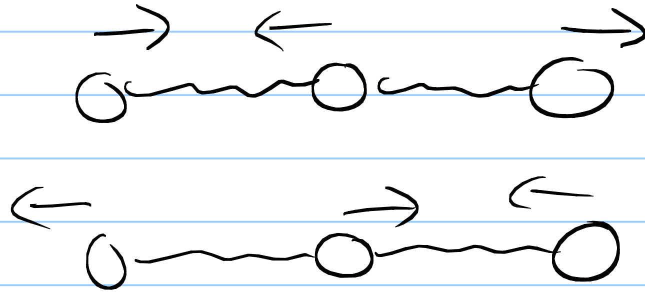
f) $\begin{pmatrix}1\\1\\1\end{pmatrix}$ bulk motion of whole lattice in one direc.

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}$$



out of phase mot.

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}$$



g) because for this mode the middle mass is at rest.

h) Static displacement
no stretching
or compression of
Lattice.

i) $\vec{X} = \sum C_N V^{(N)}$

$$\left(A - \frac{\omega^2}{\omega_0^2}\right)\vec{X} = \frac{1}{k} F$$

$$\left(A - \frac{\omega^2}{\omega_0^2}\right)\left(\sum C_N V^{(N)}\right) = \sum V^{(N)} \left(V^{(N)} \cdot \frac{F}{k}\right)$$

$$\Downarrow$$

$$\underbrace{\sum C_N A V^{(N)}} - \frac{\omega^2}{\omega_0^2} \sum C_N V^{(N)} =$$

$$\sum C_N \lambda_N V^{(N)} - \frac{\omega^2}{\omega_0^2} C_N V^{(N)} =$$

$$\sum \left(\lambda_N - \frac{\omega^2}{\omega_0^2}\right) C_N V^{(N)} = \sum V^{(N)} \left(V^{(N)} \cdot \frac{F}{k}\right)$$

$$\Rightarrow \left(\lambda_N - \frac{\omega^2}{\omega_0^2}\right) C_N = \frac{1}{k} V^{(N)} \cdot F$$

$$\downarrow$$

$$\frac{\omega_N^2}{\omega_0^2}$$

$1/m$

$$c_n = \frac{\frac{\omega_0^2}{k} v^{(n)} \cdot F}{(\omega_n^2 - \omega^2)}$$

$$\Rightarrow \quad \vec{x} = \frac{1}{m} \sum_n \frac{v^{(n)} \cdot F}{\omega_n^2 - \omega^2} v^{(n)}$$