Matrix Spaces

Text: 7.4, 7.9 Lecture Notes: 5 Lecture Slides: 1

Quote of Short Homework Four	
For a moment I lose myself, wrapped up in the pleasures of the world. I've journeyed here and there and back again but in the same old haunts I still find my friends	
	The Smashing Pumpkins : Thirty Three (1996)

1. Goals

The goal of this assignment is to review the language of linear combination. Though the algorithm of choice is row-reduction, we now concentrate on associating the pivots to the language. After this assignment the student should:

- Understand how the concepts of linear combination and independence relate to bases and dimension.

2. Objectives

To achieve the previous goals the student will meet the following objectives:

- Read section 7.4, 7.9 of the text book paying particular attention to pages 297, 300-301 and 323-325.
- Compute dependence relations for particular sets.
- Compute spans for particular sets.

3. Problems

Given,

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}. \tag{1}$$

3.1. **Row-Reduction.** Find the row echelon form of the matrix $\mathbf{V}$ whose columns are the given vectors.

3.2. **Linear Independent Sets.** Does $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a linearly independent set? What about the sets $S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $S_3 = \{\mathbf{v}_1, \mathbf{v}_2\}$, $S_4 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$, $S_5 = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3.3. **Spanning Sets.** Does S_1 span $\mathbb{R}^3$? What about S_2, S_3, S_4, S_5? If you were going to span $\mathbb{R}^3$ then which of these sets would you choose? [1]

[1]Remark: This problem is meant to demonstrate a few reoccurring points in linear algebra. The idea is that we typically work problems backwards in the sense that you start with a space of vectors, say $\mathbb{R}^n$, and ask the question,

- Given a 'vector-space' can we 'reach' every 'point' in the space.

More importantly, how can we do this with a minimal set of vectors? In this problem the vector-space is $\mathbb{R}^3$ and from calculus we know that we need only three linearly-independent vectors, typically $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, to reach every point via linear combination (aka span the space), $\mathbf{x} = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}} \in \mathbb{R}^3$. Consequently, if we choose any four vectors they must be linearly-**de**pendent, S_1, which means that some vectors point in redundant directions. However, that doesn't mean that we can pick any three vectors and still span the space, S_5. So, we have to make careful choice to take enough vectors to span the space but **not so many** that the set of vectors is linearly-**de**pendent. When we have made this careful choice we have secretly constructed a coordinate system or basis for the space. This choice is not unique, S_2, S_4.