Homework #4 Solutions:

1. Given $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

a)

$$\begin{aligned} det(A - \lambda I) &= det\left(\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}\right) = \\ &\Leftrightarrow\ 4-\lambda\left\{(1-\lambda)^2\right\} = -2(1-\lambda), \qquad \lambda = 1 \Rightarrow 0 = 0 \\ &\Leftrightarrow\ (4-\lambda)(1-\lambda) = -2 \Leftrightarrow \\ &\Leftrightarrow\ 4 + \lambda^2 - 5\lambda + 2 = 0 \Leftrightarrow \\ &\Leftrightarrow\ \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \qquad \lambda = 2, 3 \end{aligned}$$

Thus, A has three eigenvalues, $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

b)

Case $\lambda = 1$:

$$\begin{aligned} \left[\ A - \lambda I \mid 0\ \right] &= \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\Rightarrow \begin{array}{l} -2x_1 = 0 \\ 3x_1 = -x_3 \\ x_2 = \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

The basis for the eigenspace associated with $\lambda = 1$ is $B_{\lambda=1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Case $\lambda = 2$:

$$\begin{aligned} \left[\ a - \lambda I \mid 0\ \right] &= \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \\ &\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 == 1/2x_3 \\ x_2 = x_3 \\ x_3 = \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} x_3 \end{aligned}$$

The basis for this case is $B_{\lambda=2} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Case $\lambda = 3$:

$$\left[\ A - \lambda I \mid 0\ \right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \quad \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 = \text{free} \end{array} \quad \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

$$\Rightarrow \quad \text{Basis for eigenspace of A when } \lambda = 3 \text{ is } B_{\lambda=3} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2.
a) To find the eigenfunctions we find the eigenvalues and eigenvectors of A.

$$\begin{aligned} det(A - \lambda I) &= (2-\lambda)(1-\lambda) - (-2) = \lambda^2 - 4\lambda + 5 \\ \lambda &= \frac{-(-4) \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i \end{aligned}$$

Case $\lambda = 2 + i$:

$$\begin{aligned} \left[\begin{array}{c|c} A - \lambda I & 0 \end{array} \right] &= \left[\begin{array}{cc|c} 3-(2+i) & 1 & 0 \\ -2 & 1-(2+i) & 0 \end{array} \right] = \\ &= \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array} \right] \Rightarrow (1-i)x_1 + 1x_2 = 0 \end{aligned}$$

if $x \in Nul(A - \lambda I)$ then this relationship must hold.

Let $x_1 = -1 \Rightarrow x_2 = (1-i) \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1-i \end{bmatrix}$

Case $\lambda = 2 - i$: $\Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$

Thus the eigenfunctions of $\vec{x}' = A\vec{x}$ are given as,

$$\vec{x}_1(t) = \begin{bmatrix} -1 \\ 1-i \end{bmatrix} e^{(2+i)t}, \qquad \vec{x}_2(t) = \begin{bmatrix} -1 \\ 1+i \end{bmatrix} e^{(2-i)t}$$

b) Using the formula on page 354 we have the real valued general solution

$$\begin{aligned} \vec{x}(t) &= c_1\vec{y}_1(t) + c_2\vec{y}_2(t) = \\ &= c_1 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix} sin(t) \right\} e^{2t} + c_2 \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} cos(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} sin(t) \right\} e^{2t} \end{aligned}$$

3. Yes, A is diagonalizable. Why? The following process will produce four linearly independent eigenvectors.
1st: Since A is triangular we know the eigen values of A are

$$\begin{aligned} \lambda_1 &= 4 \quad \text{(With algebraic multiplicity of 2)} \\ \lambda_2 &= 2 \quad \text{(With algebraic multiplicity of 2)} \end{aligned}$$

2nd:

Case $\lambda = 4$:

$$\left[\begin{array}{c|c} A-4I & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \begin{array}{l} x_3 = 0 \\ x_1 = 2x_4 \\ x_2, x_4 = \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the basis for the eigenspace is $B_{\lambda=4} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Case $\lambda = 2$:

$$\left[\begin{array}{c|c} A-2I & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = \text{free} \\ x_4 = \text{free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ This implies that

$$P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

4.

a) Yes. The matrix is a stochastic matrix because its columns are probability vectors. It is regular since P has all non-negative entries.

b)

$$\begin{aligned} P\vec{q} = \vec{q} \;&\Leftrightarrow\; (P-I)\vec{q} = 0 \Leftrightarrow \\ &\Leftrightarrow \left[\begin{array}{cc|} -.9 & .6 \\ .9 & .6 \end{array}\right] \sim \left[\begin{array}{cc|} -.9 & .6 \\ 0 & 0 \end{array}\right] \\ &\Rightarrow \begin{array}{l} .9q_1 = .6q_2 \\ q_2 = \text{free} \end{array} \Rightarrow \vec{q} = \begin{bmatrix} 2/3q_2 \\ q_2 \end{bmatrix} = q_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \end{aligned}$$

$$\text{choose } q_2 \text{ to be } q + \frac{2}{3}q_2 = 1 \Leftrightarrow q_2 = \left(\frac{5}{3}\right) = 1 \Leftrightarrow q_2 = \frac{3}{5}$$

$$\Rightarrow \quad \vec{q} = \begin{bmatrix} 6/15 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 2/15 \\ 3/5 \end{bmatrix}$$

$$\text{Note:} \qquad P\vec{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Step 1: Diagonalize P

$$\begin{aligned}
det\left(\begin{bmatrix} .1-\lambda & .6 \\ .9 & .4-\lambda \end{bmatrix}\right) &= (.4-\lambda)(.1-\lambda) - .54 = \lambda^2 - .5\lambda - .54 + .4 = \\
&= \lambda^2 - .5\lambda - .5 \Rightarrow \lambda_1 = 1 \\
\lambda &= \frac{-(-.5) \pm \sqrt{(-.5)^2 - 4(1)(-.5)}}{2(1)} = \frac{.5 \pm 1.5}{2} = 1, -.5 \\
&= \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -.5 \end{matrix}
\end{aligned}$$

Case $\lambda = 1$: $\qquad (P - I)\vec{x} = 0 \Rightarrow x = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$

Case $\lambda = -.5$:

$$\begin{aligned}
(P - (-.5)I)\vec{x} &= 0 \Rightarrow \left[\begin{array}{cc|c} .6 & .6 & 0 \\ .9 & .9 & 0 \end{array}\right] \Rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
\text{Thus} \qquad P &= \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -3/5 & 2/5 \end{bmatrix} \cdot \frac{1}{-1} \\
\text{and} \qquad P^k &= \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2/5 \end{bmatrix} \\
\text{Thus,} \qquad \lim_{k\to\infty} P^k &= \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2/5 \end{bmatrix} = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}
\end{aligned}$$

This implies that for $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 + x_2 = 1$

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} BP^kB^{-1}x_0 = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .4(x_1 + x_2) \\ .6(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$$

4. Let $\hat{S}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

a) $\hat{S}_y^H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \hat{S}_y \Rightarrow \hat{S}_y$ is self-adjoint.

b) $det(\hat{S}_y^H - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\underline{\lambda_1 = 1}$

$$\left[\ \hat{S}_y - 1I \ \middle|\ 0\ \right] = \left[\begin{array}{cc|c} -1 & -i & 0 \\ i & -1 & 0 \end{array}\right] \Rightarrow x_1 - ix_2 = 0 \Rightarrow \begin{bmatrix} x_1 = x_2 \\ x_2 = \text{free} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} \Rightarrow \vec{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\vec{v}_1 = \frac{1}{\parallel \vec{x}_1 \parallel}\vec{x}_1 = \frac{1}{\sqrt{x^T \bar{x}}} \begin{bmatrix} -i \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{(normalized eigenvector)}$$

$\underline{\lambda = -1}$

$$\left[\ \hat{S}_y + I \mid 0\ \right] = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = ix_2 \\ x_2 = \text{free} \end{matrix} \Rightarrow \vec{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\text{Thus } A = PDP^H = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$PP^H = \begin{bmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 + 1/2 & -i/2 + i/2 \\ i/2 - i/2 & 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c)

$$\begin{aligned} A &= \lambda_1 \vec{v}_1 \vec{v}_1^H + \lambda_2 \vec{v}_2 \vec{v}_2^H = \\ &= 1 \cdot \begin{bmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} - 1 \cdot \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \\ &= \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 & i/2 \\ -1/2 & i/2 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{aligned}$$