MATH 348 - Advanced Engineering Mathematics September 12, 2008

Homework 4, Fall 2008 **Due**: September 22, 2008

Eigenvalues - Eigenvectors - Diagionalization - Spectral Decomposition - Applications

1. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

   (a) Determine the eigenvalues of $\mathbf{A}$.

   (b) Determine the eigenvectors of $\mathbf{A}$.

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

   Determine the eigenvalues and eigenfunctions associated with the system of differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A} \cdot \mathbf{x}(t)$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

   If $\mathbf{A}$ is diagonalizable, then determine $\mathbf{D}$ and $\mathbf{P}$ associated with its decomposition $\mathbf{PDP}^{-1}$. <u>Do not find $\mathbf{P}^{-1}$.</u>

4. Square matrices having columns whose entries sum to 1 are often called stochastic matrices. Those with only non-negative entries, for some power, are called *regular* stochastic matrices. Given a random process, with an initial state $\mathbf{x}_0$, the application of $\mathbf{P}$ on $\mathbf{x}_0$ discretely steps the process forward in time. That is $\mathbf{x}_{n+1} = \mathbf{P}\mathbf{x}_n = \mathbf{P}^n\mathbf{x}_0$, $n = 1, 2, 3, \ldots$. If a matrix is a *regular* stochastic matrix then there exists a steady-state vector $\mathbf{q}$ such that $\mathbf{Pq}$=$\mathbf{q}$. This vector determines the long term probabilities associated with an arbitrary inital state $\mathbf{x}_0$. The sequence of states, $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}\}$, is called a *Markov Chain*. Given the regular stochastic matrix:

$$\mathbf{P} = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}.$$

   (a) Show that the steady-state vector of $\mathbf{P}$ is $\mathbf{q} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}^{\mathrm{T}}$.

   (b) Find the matrices $\mathbf{D}$ and $\mathbf{Q}$ such that $\mathbf{P}$= $\mathbf{QDQ}^{-1}$. That is, diagonalize the matrix $\mathbf{P}$.

   (c) Show that $\lim_{n\to\infty} \mathbf{P}^n\mathbf{x}_0 = \mathbf{q}$ where $\mathbf{x}_0 = [x_1,\, x_2]^{\mathrm{T}}$ is an arbitrary vector in $\mathbb{R}^2$ such that $x_1 + x_2 = 1$.

5. Recall the Pauli Spin Matrix from homework 1,

$$\sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

   (a) Show that σ_y is self-adjoint.

   (b) Find the orthogonal diagonalization of σ_y.

   (c) Show that $\sigma_y = \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the normalized eigenvectors from part (b).