

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Briefly respond to the following:

- (a) Explain why a function defined on a finite portion of the real line has a Fourier series representation.
 Is this representation unique?

The f_{xn} on the finite domain can be Repeated
 \Rightarrow Periodic $f_{xn} \Rightarrow$ Fourier Series Rep.

This Rep. is not unique. Before
 this periodic Extension one can induce
 an even or odd symmetry, which doubles
 the width of the principle period + changes
 the series.

$$\begin{cases} f: [a,b] \rightarrow \mathbb{R} \\ f^*: \mathbb{R} \rightarrow \mathbb{R} \\ \text{such that} \\ f^*(x) = f(x), x \in [a,b] \\ f^*(x+p) = f^*(x) \\ p = b-a \end{cases}$$

- (b) Compare and contrast Fourier series with Fourier integral. Specifically:

- What is the purpose of each.
- How does one get from Fourier series to Fourier integral.
- What are similarities and differences between the linear combinations of their oscillatory modes?

$$\text{F. S.} \xrightarrow[\substack{\lim \\ L \rightarrow \infty}]{(i)} \text{F. I}$$

(i) Represents
 a periodic
 f_{xn}

Cannot Rep. f_{xn}
 which are not
 periodic

(iii) Sums $\sum_{n=-\infty}^{\infty}$
 of nodes $e^{i\omega_n x}$
 where
 $\omega_n = \frac{n\pi}{L}$

(i) Rep. f_{xn} which
 need not be periodic

(iii) Sums $\int_{-\infty}^{\infty}$
 of modes e^{-iwx}
 where $w \in \mathbb{R}$

2. (10 Points) Let,

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \pi - x, & 0 \leq x < \pi. \end{cases} \quad (1)$$

Find the Fourier series representation of f .

$$a_0 = \frac{1}{2\pi} \int_0^\pi (\pi - x) dx = \frac{1}{2\pi} \left[\pi x - \frac{1}{2} x^2 \right]_0^\pi = \frac{1}{2\pi} \left[\pi^2 - \frac{\pi^2}{2} \right] = \frac{\pi}{4}$$

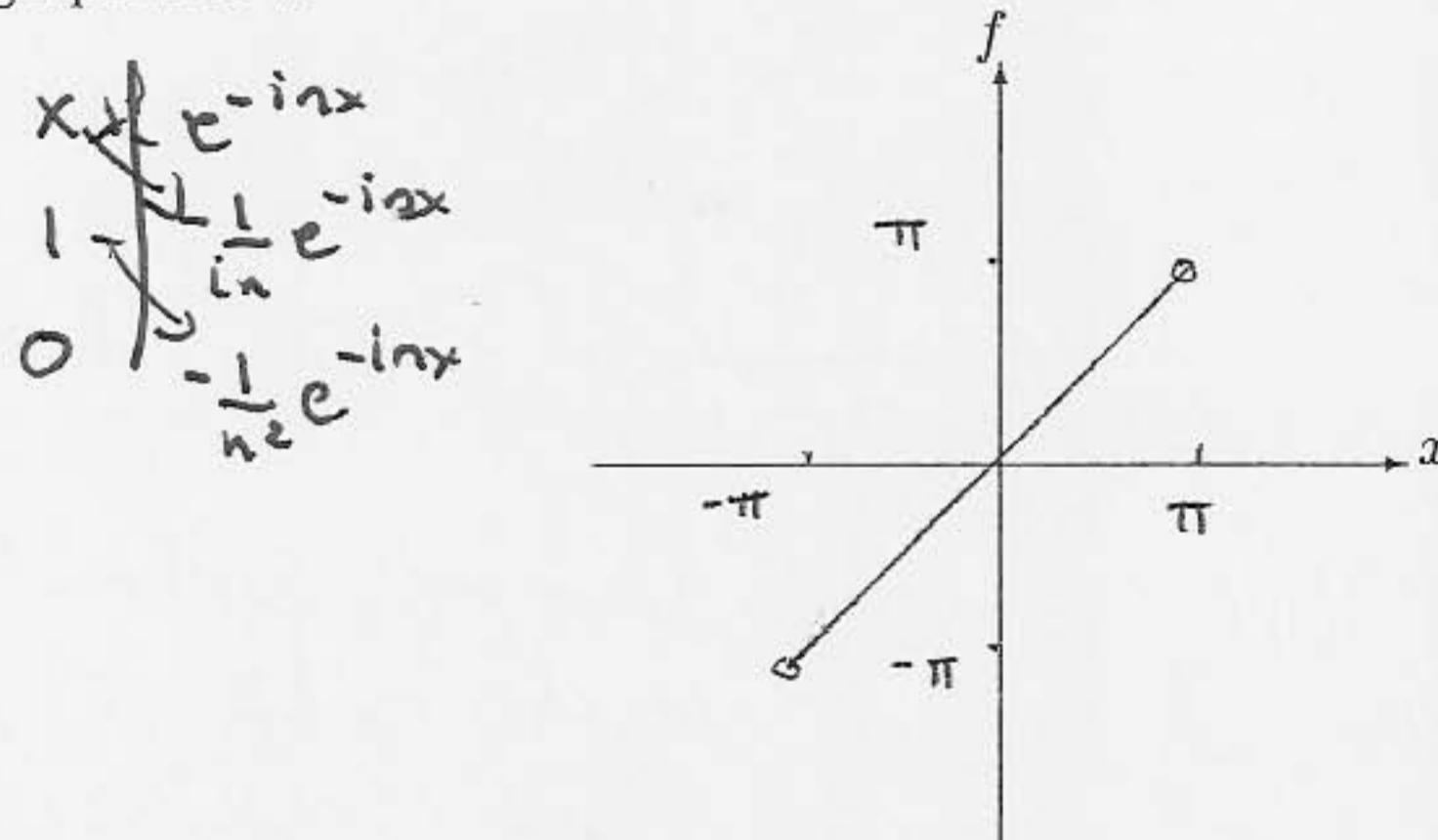
$$a_n = \frac{1}{\pi} \int_0^\pi (\pi - x) \cos(nx) dx = \frac{1}{\pi} \left[-\frac{\cos(nx)}{n^2} \right]_0^\pi = \frac{-1}{\pi} \left[(-1)^n - 1 \right] \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^\pi (\pi - x) \sin(nx) dx = \frac{1}{\pi} \left[-(\pi - x) \frac{\cos(nx)}{n} \right]_0^\pi = \frac{1}{n}$$

$$\begin{array}{c|cc|c} u & dv_1 & dv_2 \\ \hline \pi - x & \cos(nx) & \sin(nx) \\ -x & +\frac{\sin(nx)}{n} & -\frac{\cos(nx)}{n} \\ 0 & -\frac{\cos(nx)}{n^2} & -\frac{\sin(nx)}{n^2} \end{array}$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \left[\pi^2 - \frac{\pi^2}{2} \right] + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{\pi n^2} \cos(nx) + \frac{\sin(nx)}{n}$$

3. (10 Points) Find the complex Fourier series representation of the function given on its principle period in the graph below.



$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{inx} dx = \\ &= \frac{1}{2\pi} \left[-\frac{x e^{-inx}}{in} + \frac{e^{-inx}}{in^2} \right]_{-\pi}^{\pi} = \\ &= \frac{1}{2\pi} \left[-\frac{\pi e^{-i\pi}}{in} - \frac{(-\pi) e^{i\pi}}{in} \right] = \\ &= -\frac{(-1)^n}{in}, \quad n \neq 0 \end{aligned}$$

$$f(x) = \sum_{n=-\infty, n \neq 0}^{\infty} i \frac{(-1)^n}{n} e^{inx}$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx = 0$$

4. (10 Points) Let $f(x) = \pi$ for $x \in [0, L]$. Find both the Fourier cosine and sine half-range expansions of f .

Cosine Expansion implies $f(x) = \pi$, $x \in \mathbb{R}$

$$\Rightarrow b_n = 0, a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos\left(\frac{n\pi}{L}x\right) dx = 0, a_0 = \frac{1}{2L} \int_{-\pi}^{\pi} \pi dx = \pi$$

$$\Rightarrow f_c(x) = \pi$$

Sine Expansion:

$$a_0 = a_n = 0, b_n = \frac{2}{\pi} \int_0^{\pi} \pi \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{\pi} \cdot 2 \left[-\frac{\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}} \right]_0^{\pi} =$$

$$= -\frac{2}{n} [\cos(n\pi) - 1] = b_n$$

$$\Rightarrow f_s(x) = \sum_{n=1}^{\infty} 2 \left[\frac{(-1)^n}{n} \right] \sin\left(\frac{n\pi}{L}x\right)$$

5. (10 Points)

(a) Let $\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n)$. Find the inverse Fourier transform of \hat{f} .

$$f(x) = \mathcal{F}^{-1}\{\hat{f}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n) e^{inx} d\omega =$$
~~$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(\omega - n) e^{inx} d\omega = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}$$~~

(b) Suppose that $c_n = i \frac{(-1)^n}{n}$ for $n \neq 0$ and $c_0 = 0$. Graph the function $f(x) = \mathcal{F}^{-1}\{\hat{f}\}$ below.

Since there
are the coeff.
from 3
we have
that

$$f(x) = \frac{1}{\sqrt{2\pi}} f(x)$$

Problem 4

Problem 3

