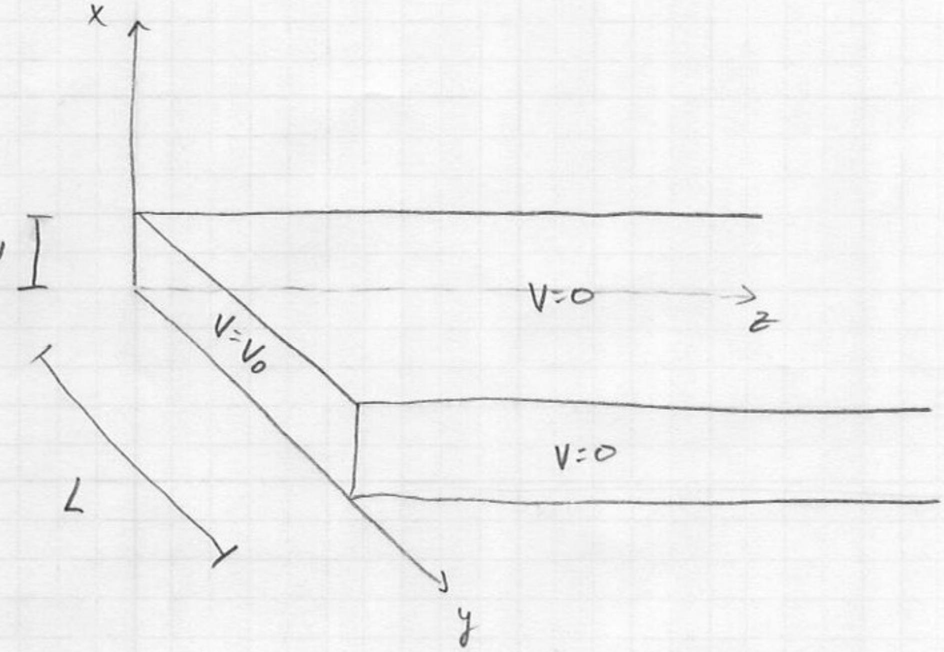
Phys 361 Recitation - Separation of variables

1) a)



The equation that fully describes electrostatics is Poisson's equation:

$$-\nabla^2 V = \rho/\varepsilon_0$$

But if we're searching for the potential inside the hollow pipe and nowhere else, $\rho = 0$ everywhere we're looking and Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0$$ or, in Cartesian,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad (1)$$

b) The cap at $z=0$ breaks translational symmetry in that direction, and x and y don't behave any better. We're solving our equations in all three variables.

c) I'll guess that $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

and I'll substitute that into (1).

$$\frac{d^2X}{dx^2}YZ + X\frac{d^2Y}{dy^2}Z + XY\frac{dZ}{dz} = 0$$

Now I'll divide by XYZ:

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\mathrm{I}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\mathrm{II}} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\mathrm{III}} = 0$$

d) Note that I, II, and III are each functions of single variables. That requires that each be equal to some constant.

How can I be so sure? Well, suppose $\frac{1}{X}\frac{d^2X}{dx^2}$ did vary with x. II and III definitely don't vary with x, so if I fiddle with x, I'll change I without changing II and III, and I'll change their sum. But

I + II + III has to equal zero for all x, y, and z for

us to satisfy Laplace's equation. So we'd have a contradiction.

The only way to obtain a solution everywhere is to have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \kappa_1^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = \kappa_2^2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = \kappa_3^2$$

for some kappas such that $\boxed{\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 0}$