MATH-332: Linear Algebra Chapter: 07

Linear Equations in Linear Algebra

Section 1.7: Linear Independence

pgs. 65-72 *June 22, 2009*

Lecture: Linear Independence

Topics:
Linear Combinations
Linear Independence
Characterizations of Linearly Dependent Sets

Problems
Prac: 1 - 4
Prob: 9, 15, 17, 19, 21, 27

This is one of the most important sections in the text. Many other linear algebra textbooks leave this material for much later when it hinders more than it helps. The author can do this by initially highlighting the four-fold description of a linear system. [1] Particularly, the vector description found in section 1.3 pushes the concept of a *linear combination* to the forefront and this concept can now be used to bread the notion of *linear independence*.

The idea of linear independence of vectors is one of the most fundamental concepts in mathematics. A great example is found in differential equations. Given,

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \tag{1}$$

has a solution that can be described in terms of the linear combination,

$$\mathbf{Y}(t) = k_1\mathbf{v}_1 e^{\lambda_1 t} + k_2\mathbf{v}_2 e^{\lambda_2 t}, \tag{2}$$

so long as the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. [2] This theme is seen in more general settings where the prescription is this:

1. Determine what 'vector-space' you are working with.[3]
2. Determine a linearly independent set of vectors, which spans the vector space. [4]
3. Write down arbitrary vectors from this space as linear combinations of the linearly independent vectors.

[1]Recall that this is:

1. The linear system itself.
2. $\mathbf{Ax} = \mathbf{b}$
3. $[\mathbf{A}|\mathbf{b}]$
4. $\sum_{i=1}^{n} x_i\mathbf{a}_i = \mathbf{b}$

[2]In a more geometric setting we can say that the differential equation defines a two-dimensional solution space (this was what we called phase space) and that needs two linearly independent vectors/solutions to span it. Arbitrary solutions can then be constructed using linear combinations of these 'basis' vectors.

[3]A vector-space is exactly what the name implies, a space full of vectors. The term space implies that we have a collection of vectors together with some sort of algebra, while the term vector can mean quite a lot of things. See chapter 4 if you just can't wait.

[4]Linearly independent vectors are nice but orthogonal vectors are even better. One can quickly see that the standard basis vectors of $\mathbb{R}^2$ are linearly independent and thus any vector from the plane can take the form $\mathbf{x} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}$, but since $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ things, we will find, are even better.

4. Find the weights/coefficients/co-ordinates of the linear combination through some sort of algorithm. [5]

Now you have a general technique that can be used to 'write down' solutions to 'problems.' Well, at least up to finding some coefficients, but often much can be said without finding these coefficients. [6]

Section Goals

- Understand the connection between linear independence of vectors and trivial solutions to homogeneous equations.
- Characterize linear dependence in terms of linear combinations of vectors.

Section Objectives

- Define linear combination and linear independence of vectors.
- Present and prove theorems 7,8,9 from pages 68-69, which characterizes linearly dependent/independent sets.

[5] We will typically use the row-reduction algorithm, but there are cases when this is not a useful tool.

[6] Consider the differential equation $y'' + y = 0$ a quick check of the solution $y(t) = k_1 \sin(t) + k_2 \cos(t)$ shows the oscillations and frequency of oscillations. Finding k_1, k_2 only tells you the amplitudes of oscillation, which may not be as important as its other features.