per mile and there are $u_0\tau$ miles, then $\rho_0 u_0 \tau$ is the number of cars passing the observer in τ hours. Thus the number of cars *per hour* which we have called the traffic flow, q, is

$$q = \rho_0 u_0.$$

Although this has been derived from an oversimplified case, we will show that this is a fundamental law, the

$$\text{traffic flow} = (\text{traffic density})(\text{velocity field}).$$

If the traffic variables depend on x and t, i.e., $q(x, t)$, $\rho(x, t)$, $u(x, t)$, then we will still show that

$$q(x, t) = \rho(x, t)u(x, t). \quad (59.1)$$

An easy way to show this is to consider the number of cars that pass $x = x_0$ in a *very small* time Δt, i.e., between t_0 and $t_0 + \Delta t$. In that small time the cars have not moved far and hence (if ρ and u are continuous functions of x and t) $u(x, t)$ and $\rho(x, t)$ can be approximated by constants, their values at $x = x_0$ and $t = t_0$. In a small time Δt, the cars that occupy a short space, approximately $u(x, t)\Delta t$, will pass the observer, as shown in Fig. 59-3. The number of cars passing is approximately $u(x, t)\Delta t\, \rho(x, t)$. The traffic flow is given by equation 59.1. Thus the results for constant u and ρ do not need modification for nonuniform $u(x, t)$ and $\rho(x, t)$. Consequently, the three fundamental traffic variables, density $\rho(x, t)$, velocity $u(x, t)$ and flow $q(x, t)$, are related by equation 59.1.

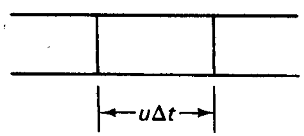

Figure 59-3 Approximate distance a car travels in Δt hours.

EXERCISES

59.1. For traffic moving at 10 m.p.h. (16 k.p.h.) such that cars are one car length behind each other, what is the traffic flow?

59.2. Suppose that at position x_0 the traffic flow is known, $q(x_0, t)$, and varies with time. Calculate the number of cars that pass x_0, between $t = 0$ and $t = t_0$.

59.3. In an experiment the total number of cars that pass a position x_0 after $t = 0$, $M(x_0, t)$, is measured as a function of time. Assume this series of points has been smoothed to make a continuous curve.
(a) Briefly explain why the curve $M(x_0, t)$ is increasing as t increases.
(b) What is the traffic flow at $t = \tau$?

60. Conservation of the Number of Cars

In this section, we formulate a deterministic model for traffic flow. Suppose that the density and the velocity field are known initially for a highway of infinite length. Can we predict the densities and velocities at future times? For example, if a traffic light turns red and shortly later green, then can the pattern of traffic be predicted?

We can consider the two fundamental traffic variables to be $\rho(x, t)$ and $u(x, t)$ (since $q = \rho u$ was demonstrated in the previous section). However, suppose we knew the initial traffic density ($\rho(x, 0)$) and the traffic velocity field for all time ($u(x, t)$). Then the motion of each car satisfies the following first order differential equation:

$$\frac{dx}{dt} = u(x, t) \quad \text{with} \quad x(0) = x_0.$$

Solving this equation (which at least can be accomplished numerically using a computer) would determine the position of each car at later times. Consequently at later times, we could calculate the density (although this calculation may be difficult; it would involve deciding what measuring interval to use). Thus, the traffic density at future times can be calculated knowing the traffic velocity (and the initial density).

We want to determine how the density can be calculated easily if the velocity is known (later we will solve problems in which the velocity also isn't known). By following each car we insisted that the number of cars stays the same. However, the traffic variables density, velocity, and flow were introduced so that we wouldn't have to follow individual cars. Let us now instead try to "conserve" cars, but do so using these traffic *field* variables.

On some interval of roadway, between $x = a$ and $x = b$, as shown in Fig. 60-1, the number of cars N is the integral of the traffic density:

$$N = \int_a^b \rho(x, t)\, dx. \quad (60.1)$$