Note Title 11/7/2006

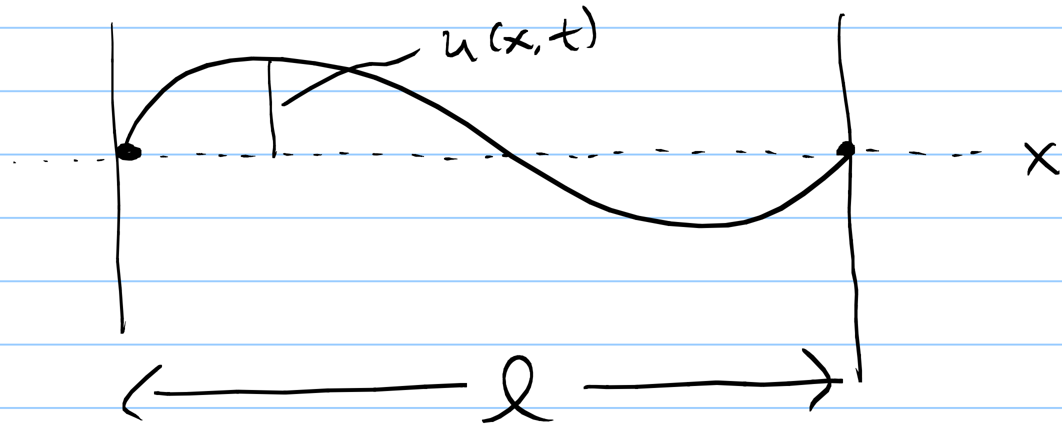
11/8/06

Review of Separation of Variables in Cartesian coordinates.

Eg.

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \underbrace{\frac{\partial^2 u(x,t)}{\partial x^2}}_{\text{Laplacian in 1D}}$$

This models the vibrations of a string clamped at the ends



Boundary conditions

$$u(0,t) = u(\ell,t) = 0$$ clamped end

initial conditions

$$u(x,0) = u_0(x)$$
$$\frac{du}{dt}(x,0) = v_0(x)$$

Guess suppose we can find a solution of the form:

$$u(x,t) = X(x)\,T(t)$$ not obviously true!

Give it a try and see if it works

prime = space deriv.

dot = time deriv.

$$\frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial u}{\partial t} = X\dot{T}$$

So

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\Rightarrow \quad X(x)\ddot{T}(t) = c^2 X''(x)T(t)$$

Always divide by XT on Both sides

$$\Rightarrow \quad \frac{X\ddot{T}}{XT} = c^2 \frac{X''T}{XT}$$

$$\Rightarrow \quad \frac{\ddot{T}}{T} = c^2 \frac{X''}{X}$$

depends only on t

depends only on x

How?

only if

$$\frac{\ddot{T}}{T} = c^2 \frac{X''}{X} = \text{constant}$$

This constant must have dimensions $\left[\frac{1}{\text{time}}\right]^2$

so it's a frequency squared.

$$\frac{\ddot{T}}{T} = c^2 \frac{X''}{X} = -\omega^2$$

minus sign will be expl. later

$$\ddot{T} + \omega^2 T = 0$$

$$X'' + \frac{\omega^2}{c^2} X = 0$$

$\Rightarrow \quad T(t) = A\cos(\omega t) + B\sin(\omega t)$

$\Rightarrow \quad X(x) = C\cos\left(\frac{\omega}{c}x\right) + D\sin\left(\frac{\omega}{c}x\right)$

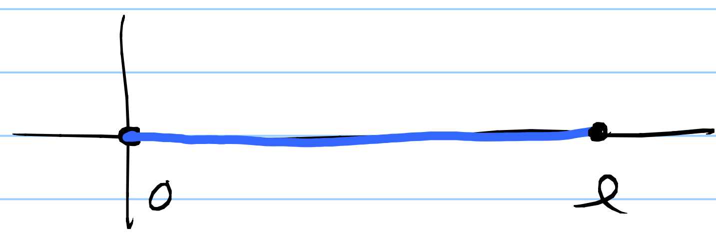
Apply B.C.

$X(0) = 0 \Rightarrow C = 0$

$X(\ell) = 0 \Rightarrow \boxed{\frac{\omega \ell}{c} = N\pi}$

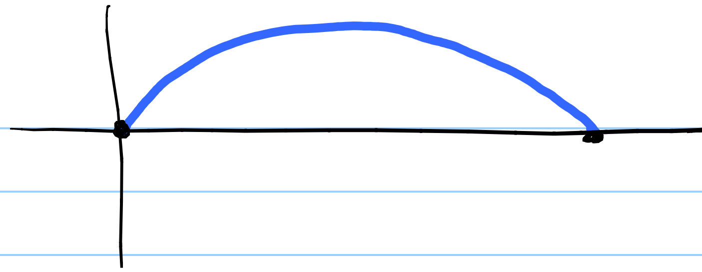
so $X(x) = D\sin\left(\frac{N\pi}{\ell}x\right)$

$N = 0$



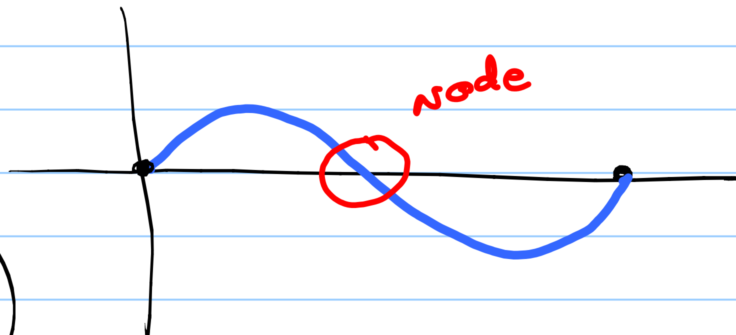
$\omega = 0$

$N = 1$



$\omega = \frac{c\pi}{\ell}$

$N = 2$



$\omega = \frac{2c\pi}{\ell}$

$$\boxed{\omega_N = \frac{N\pi c}{\ell}}$$

notice that $\omega \uparrow$ as $\underline{c} \uparrow$
stiffer string

and $\omega \downarrow$ as $\underline{\ell} \uparrow$
longer string

The spatial solutions

$$X_N(x) = D \sin(N\pi x/\ell)$$

are called modes

Now the time equation

$T(t) = A\cos(\omega t) + B\sin(\omega t)$

The BC gives us a quantization condition on frequency.

$$X(\ell) = 0 \Rightarrow \frac{\omega \ell}{c} = n\pi$$

$$\Rightarrow \omega = \frac{n\pi c}{\ell}$$

Lets call this ω_n to emphasize the discretness

$$\boxed{\omega_n = \frac{n\pi c}{\ell}}$$

So

$$T(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

$$u(x,t) = X(x)\,T(t)$$

recall $X(x) = D\sin\left(\frac{n\pi x}{\ell}\right)$

$$= D\sin\left(\frac{\omega_n}{c}x\right)$$

So

$$u(x,t) = \underbrace{AD}_{A}\cos(\omega_n t)\sin\left(\frac{\omega_n}{c}x\right) + \underbrace{BD}_{B}\sin(\omega_n t)\sin\left(\frac{\omega_n}{c}x\right)$$

$$u(x,t) = A\cos(\omega_n t)\sin\left(\frac{\omega_n}{c}x\right) + B\sin(\omega_n t)\sin\left(\frac{\omega_n}{c}x\right)$$

Does this satisfy the initial conditions?

$$u(x,0) = A\sin\left(\frac{\omega_n}{c}x\right)$$

not in general

Let

$u_1(x,t) = A_1 \cos(\omega_1 t) \sin\left(\frac{\omega_1}{c} x\right) + B \sin(\omega_1 t) \sin\left(\frac{\omega_1}{c} x\right)$

$u_2(x,t) = A_2 \cos(\omega_2 t) \sin\left(\frac{\omega_2}{c} x\right) + \cdots$

Since the PDE

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

is Linear

$$\frac{\partial^2}{\partial t^2}\left[u_1 + u_2\right] = c^2 \frac{\partial^2}{\partial x^2}\left[u_1 + u_2\right]$$

$u_1 + u_2$ is also a solution

But now we have 4 const. to help us satisfy the Initial conditions

Keep going ....

Let

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{\omega_n}{c}x\right)\cos(\omega_n t) + B_n \sin\left(\frac{\omega_n}{c}x\right)\sin(\omega_n t)$$

Now try the I. Cond.

$$\boxed{u(x,0) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{\omega_n}{c}x\right) = u_0(x)}$$

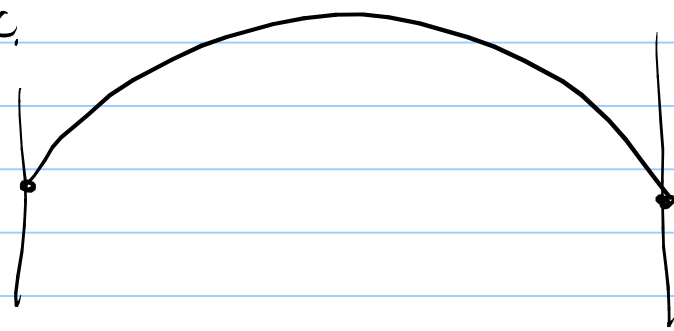
i.e. we can satisfy one of the I.C. if we can represent $u_0(x)$ as a Fourier Series

How about $\left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0$

$$\boxed{V_0(x) = \sum_{N=0}^{\infty} \omega_N B_N \sin\left(\frac{\omega_N}{c} x\right)}$$

Suppose we pluck a string

Ex.



snapshot at $t = 0$

$$\underline{u(x,0) = A_1 \sin\left(\omega_1/_c\, x\right)}$$

$\underline{N=1 \text{ mode}}$

$$= A_1 \sin(\pi x/\ell)$$

$$= \sum_{N=0}^{\infty} A_N \sin\left(\frac{\omega_N}{c} x\right)$$

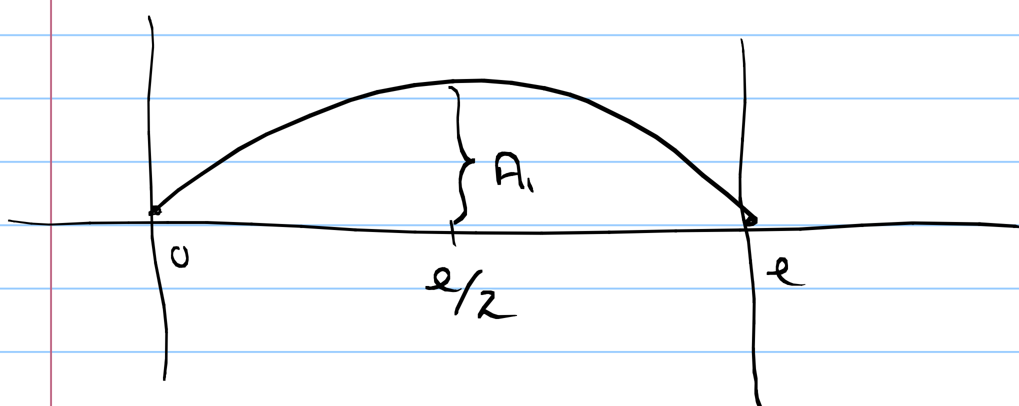
$$\Rightarrow A_N = \begin{cases} A_1 & \text{if } N=1 \\ 0 & \text{otherwise} \end{cases}$$

$$v(x,0) = 0 \Rightarrow B_N = 0$$

$$\Rightarrow \quad u(x,t) = A_1 \sin\left(\frac{\omega_1}{c}x\right)\cos(\omega_1 t)$$

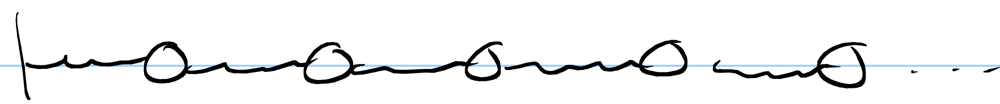
$$\boxed{u(x,t) = A_1 \sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi c}{\ell}t\right)}$$

complete solution



$$u(\ell/2, 0) = A_1 \sin(\pi/2) = A_1$$

So A_1 is specified by how far you pull the string back.



$\lim_{N\to\infty}$ discrete equations Newton's 2nd

$\Rightarrow$ wave equation

$$\boxed{u(x, t=0) = \sum A_n \sin(\omega_n/c \; x)}$$