MATH 225 - Differential Equations May 15, 2008
Homework 3, Field 2008 **Due**: May 20, 2008

QUALITATIVE ANALYSIS - EXISTENCE AND UNIQUENESS - PHASE LINE

1. Section 1.3 of the text, problems 8, 10, 15.

2. Consider the following logistics models for population growth,

$$\frac{dP}{dt} = f_H(P) = kP\left(1 - \frac{P}{N}\right) - H \tag{1}$$

$$\frac{dP}{dt} = f_\alpha(P) = kP\left(1 - \frac{P}{N}\right) - \alpha P \tag{2}$$

where k, N, M, H, α are the growth rate, carrying capacity, minimum threshold, harvesting and harvesting rate parameters respectively.

   (a) For (1) let $k = N = 2$ and $H = 0.5$. Using HPGSOLVER, with the domain $t \in (-3, 5)$ and $y \in (-1, 5)$, to plot the slope field and solutions associate the initial conditions $(0, .25)$, $(0, .5)$ $(0, 3)$ and discuss the long term behavior for each solution.

   (b) For (2) let $k = N = 2$ and $\alpha = 0.5$. Using HPGSOLVER, with the domain $t \in (-3, 5)$ and $y \in (-1, 5)$, to plot the slope field and solutions associate the initial conditions $(0, .125)$, $(0, .25)$ $(0, 5)$ and discuss the long term behavior for each solution.

   (c) Compare these two harvesting models, which would you use to harvest a population where P cannot be exactly known? What if you could always know exactly the population P, which would you use then?

3. Assuming f satisfies the hypotheses of the Uniqueness Theorem and that $y_1(t) = 4+t+3t^2$ and $y_2(t) = \dfrac{1}{t^2 + 2t + 3}$ are solutions to $\dfrac{dy}{dt} = f(t, y)$. What can you conclude about the solution to $\dfrac{dy}{dt} = f(t, y)$ where $y(0) = \dfrac{1}{2}$ for all $t \in \mathbb{R}$?

4. Given $\dfrac{dy}{dt} = y(y - 2)(y - 4)$,

   (a) Sketch the phase line and classify all equilibrium points.

   (b) Next to your phase line, sketch the solutions satisfying the initial conditions $y(0) = -1$, $y(0) = 1$, $y(0) = 3$, and $y(0) = 5$.

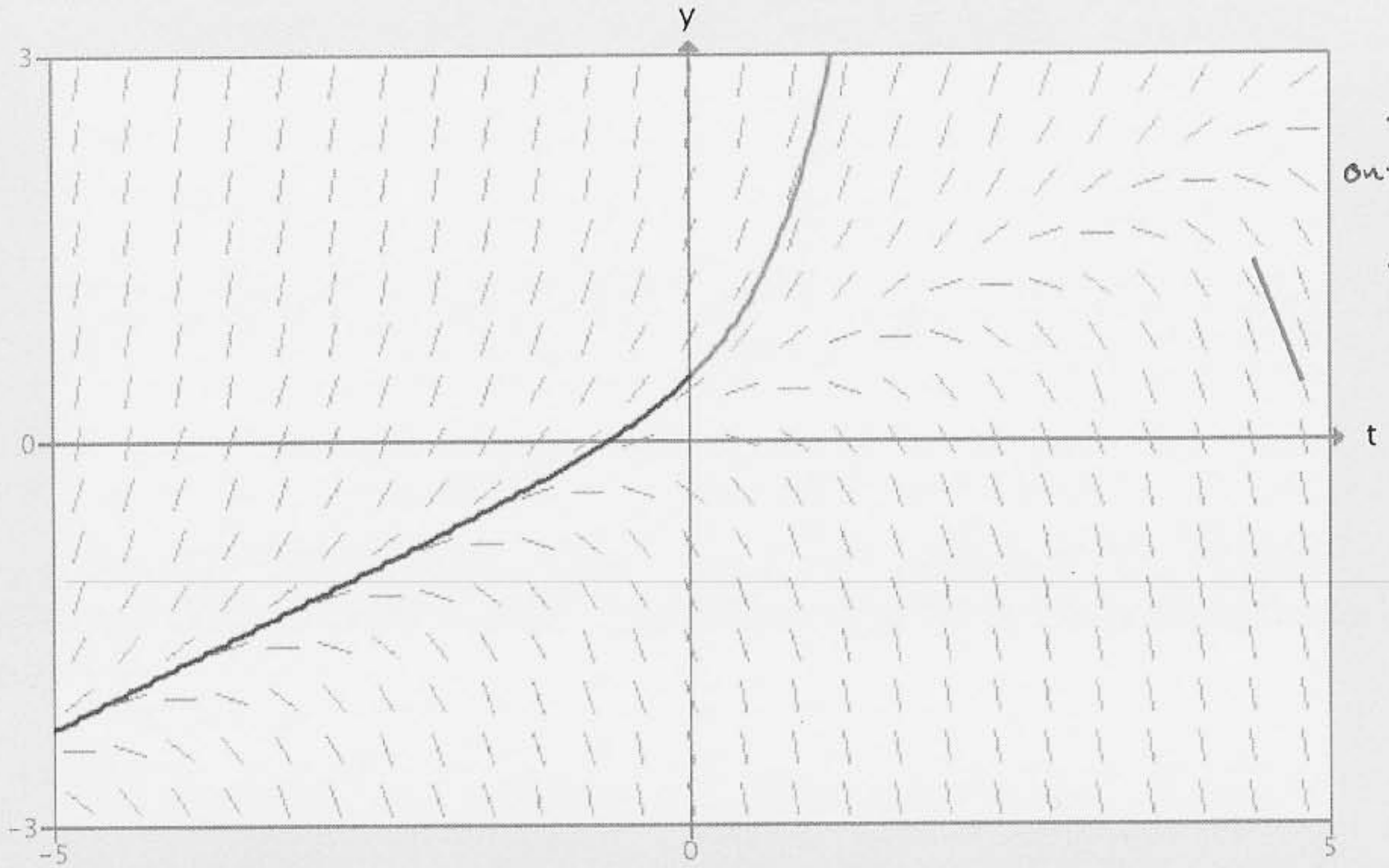
   (c) Describe the long-term behavior of the solution that satisfies the initial condition $y(0) = 1$.

5. Given $\dfrac{dy}{dt} = \sin(y^2)$,

   (a) Sketch the phase line and classify all equilibrium points. (You cannot sketch the entire phase line in this case, but show at least 5 equilibrium points, including both positive and negative y-values.)

   (b) Next to your phase line, sketch the solutions satisfying the initial conditions $y(0) = \sqrt{\dfrac{\pi}{2}}$, and $y(0) = \sqrt{\dfrac{3\pi}{2}}$.

   (c) Describe the long-term behavior of the solution that satisfies the initial condition $y(0) = \sqrt{\dfrac{\pi}{2}}$.



The soln which is shown on the graph follows a traj. which diverges to ∞ as $t \to \infty$.

Clear | Hide Field | Equations | Runge Kutta 4 | Draw Solutions | Draw Slopes

t= 4.6

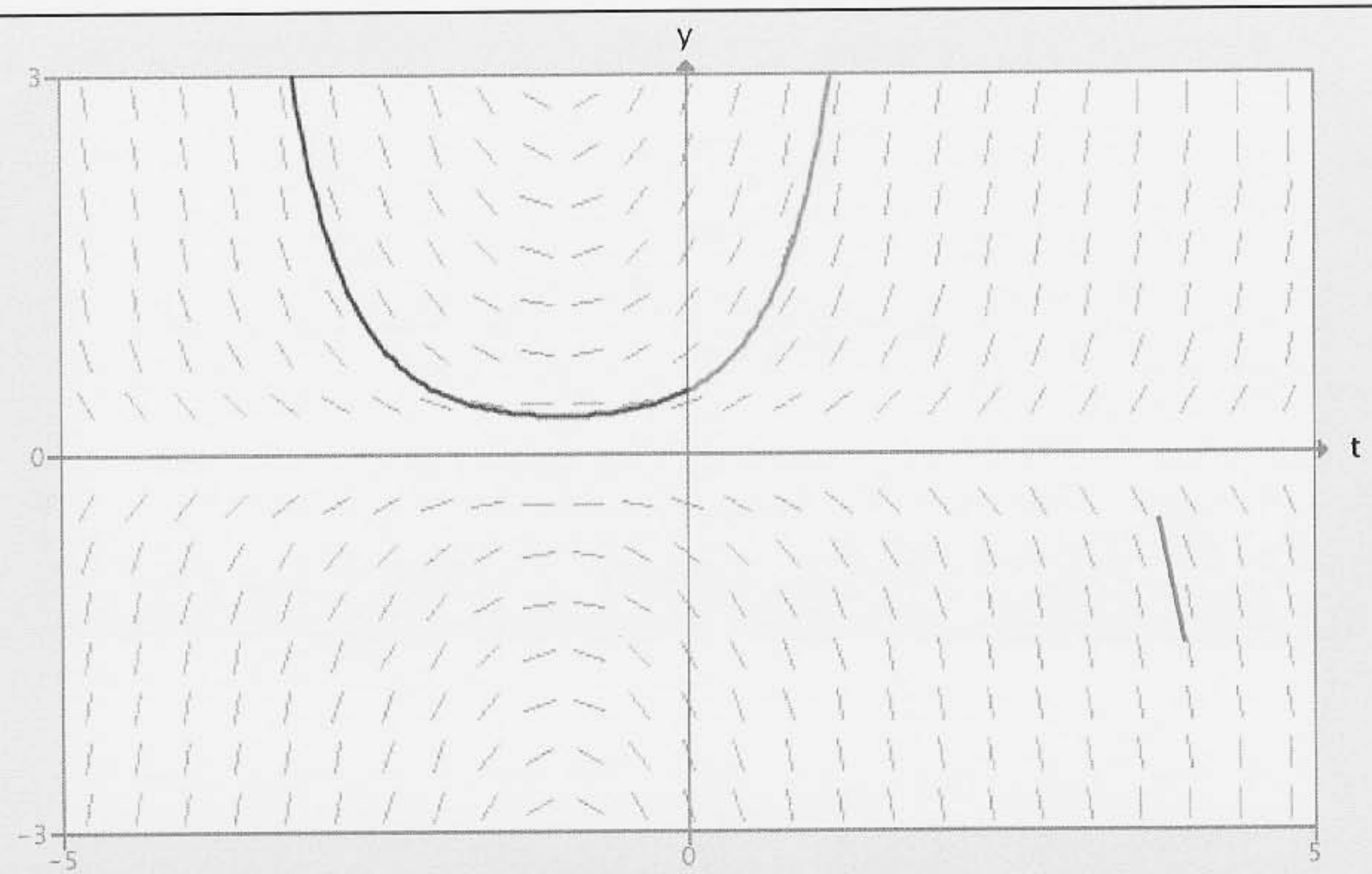
y= 0.92 dy/dt= 1.84

dy/dt = 2*y-t

min y -3 max y 3 y_0 0.5

min t -5 max t 5 t_0 0

Reset | Zoom Out | Zoom In | Solution | delta t 0.05



See the
previous
Statement.

Clear | Hide Field | Equations | Runge Kutta 4 | Draw Solutions | Draw Slopes

t= 3.86

y= -1.02   dy/dt= -1.02

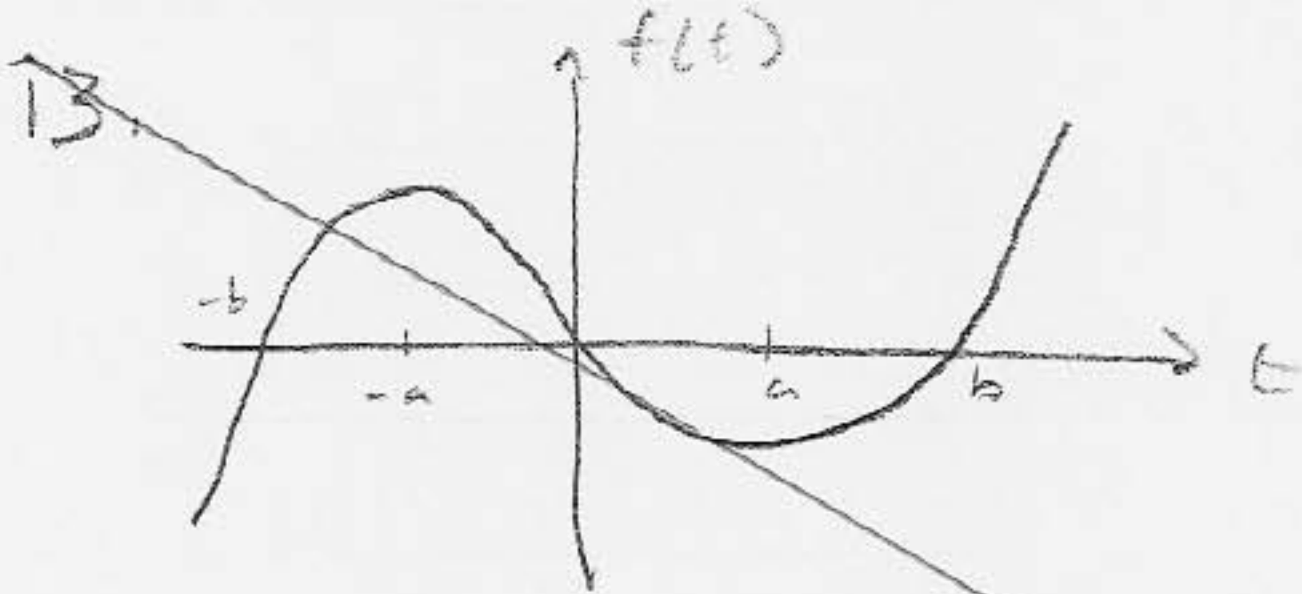
dy/dt = (t+1)*y

min y -3   max y 3 y_0 0.5

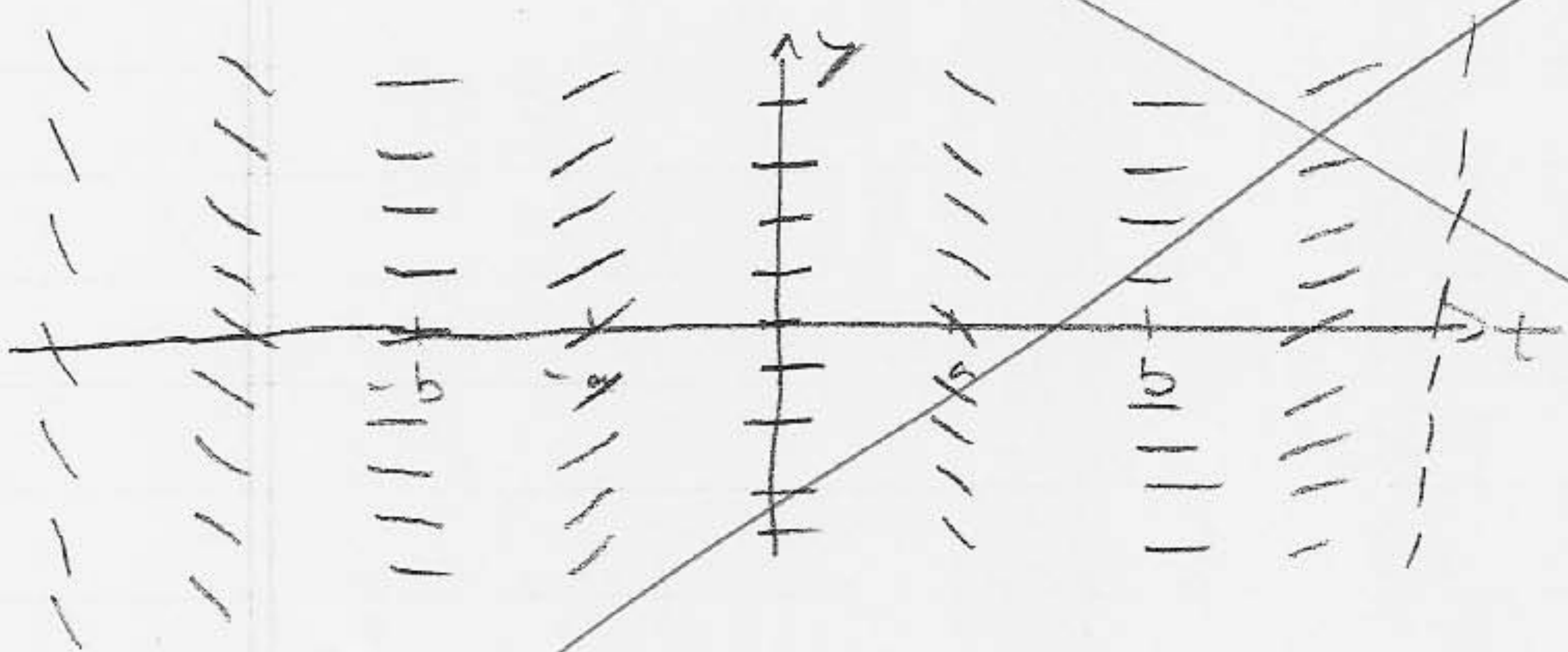
min t -5   max t 5 t_0 0

Reset | Zoom Out | Zoom In | Solution | delta t 0.05

13.



$\frac{dy}{dt} = f(t)$



t	$f(t)$	$\frac{dy}{dt}$
0	0	0
a	~ -1	~ -1
b	0	0
$-a$	~ 1	~ 1
$-b$	0	0

15. (a) Does not depend on y $\left(\frac{dy}{dt} = f(t)\right)$

$t = 1$, $\frac{dy}{dt} = 0$

$t = 0$, $\frac{dy}{dt} > 0$

$\boxed{(iv)}$

(b) Does depend on $t + y$ $\left(\frac{dy}{dt} = f(t,y)\right)$

$y = 1$, $\frac{dy}{dt} = 0$ for all t

$\boxed{(vii)}$

(c) Does not depend on t. $\left(\frac{dy}{dt} = f(y)\right)$

$y = 1$, $\frac{dy}{dt} = 0$

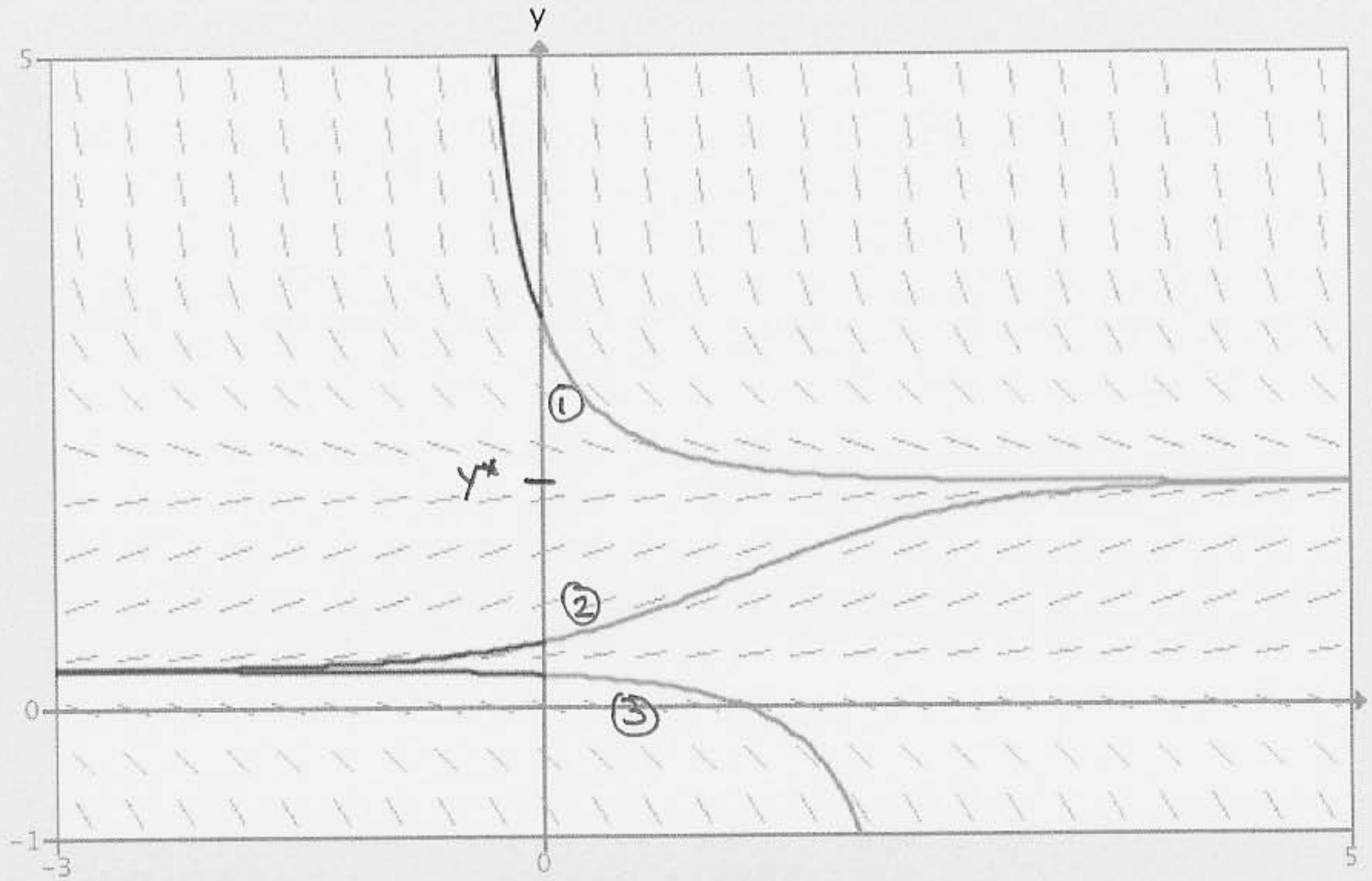
$y = -1$, $\frac{dy}{dt} = 0$

$y = 0$, $\frac{dy}{dt} < 0$

$\boxed{(viii)}$

$\boxed{(vi)}$ (d) Does depend on $t + y$ $\left(\frac{dy}{dt} = f(t,y)\right)$

as $t \to \infty$, $\frac{dy}{dt} \to \infty$ as $t \to -\infty$, $\frac{dy}{dt} \to \infty$



as $t \to \infty$
Sol$\underline{n}$ 1, 2 tend to an Equilibrium value y^*.

3 Decays to $y=0$ after this the Results are nonphysical.

Clear | Hide Field | Equations | Runge Kutta 4

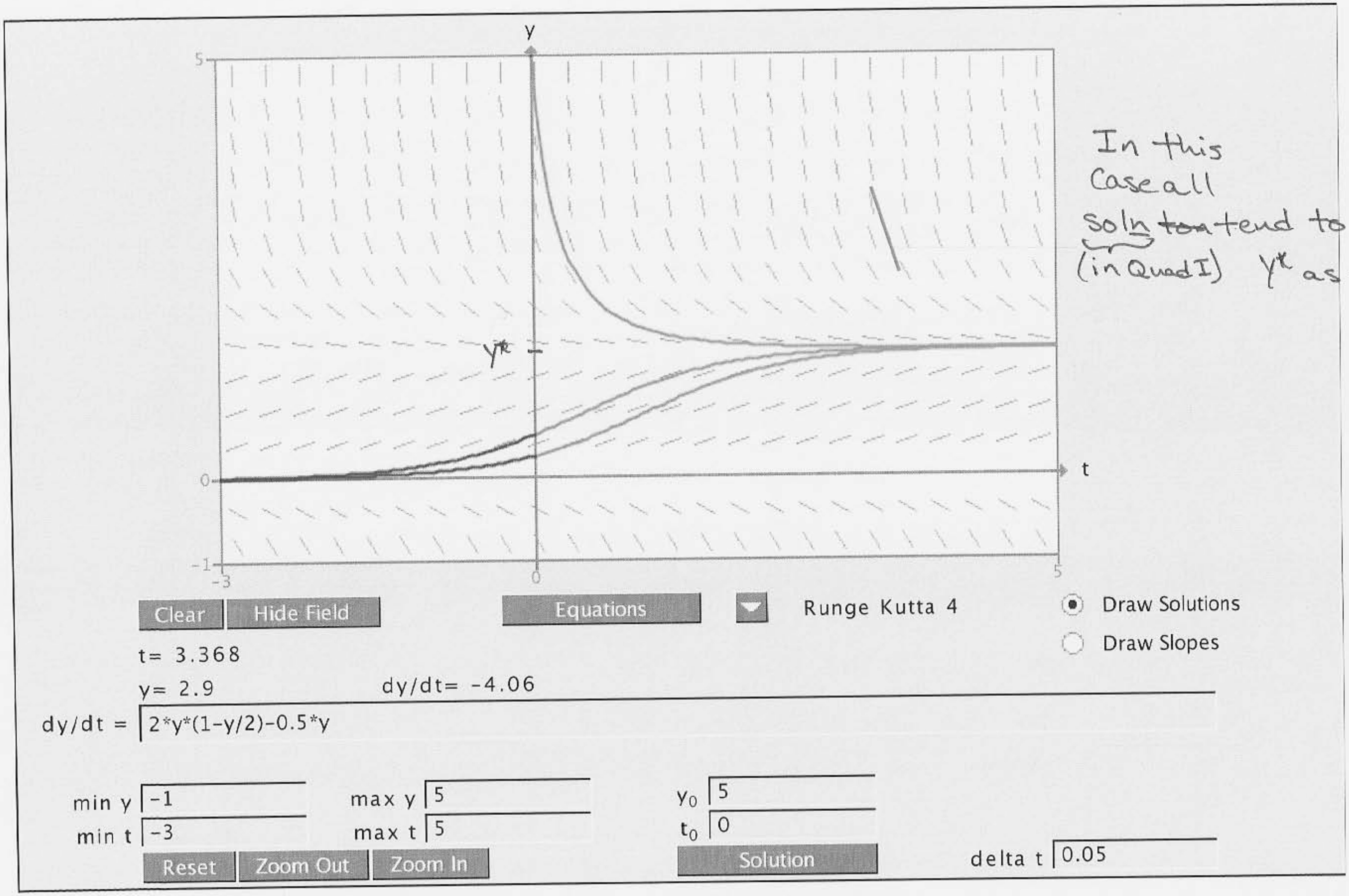
Draw Solutions
Draw Slopes

t= 0
y= 0.5 dy/dt= 0.25

dy/dt = 2*y*(1-y/2)-0.5

min y -1 max y 5 y_0 0.5
min t -3 max t 5 t_0 0

Reset | Zoom Out | Zoom In | Solution | delta t 0.05



In this case all $\underbrace{\text{soln}}_{\text{(in Quad I)}}$ ~~ton~~ tend to y^* as $t \to \infty$.

c) If P is known then it is safe to harvest at a constant value H. So long as $H < P$. However, if P is unknow it is safer to harvest at a Rate prop. to the assumed total pop. So that $P \to 0$.

(b) is more accurate

4. If $y_1(t) = 4 + t + 3t^2$ and $y_2(t) = \dfrac{1}{t^2 + 2t + 3}$ are solutions to $\dfrac{dy}{dt} = f(t, y)$, what can you conclude about the solution to $\dfrac{dy}{dt} = f(t, y)$ where $y(0) = \dfrac{1}{2}$? Assume f satisfies the hypotheses of the Uniqueness Theorem, that is $f(t, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous functions in the entire ty-plane.

$y_1(t) = 4 + t + 3t^2$ is continuous

$y_2(t) = \frac{1}{t^2+2t+3}$ is continuous

$t^2 + 2t + 3 \neq 0$ for any real t

$\left(t = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}\, i\right)$

$y_1(0) = 4$, $y_2(0) = \frac{1}{3}$

$y_2(0) < y(0) < y_1(0)$

$$\boxed{\begin{gathered} y_2(t) < y(t) < y_1(t) \\ \frac{1}{t^2+2t+3} < y(t) < 4 + t + 3t^2 \end{gathered}}$$

since solutions exist + are unique, they cannot cross.

MATH225, Spring 2008 Worksheet 4 (1.6, 1.7, 1.8)

Name: Solutions

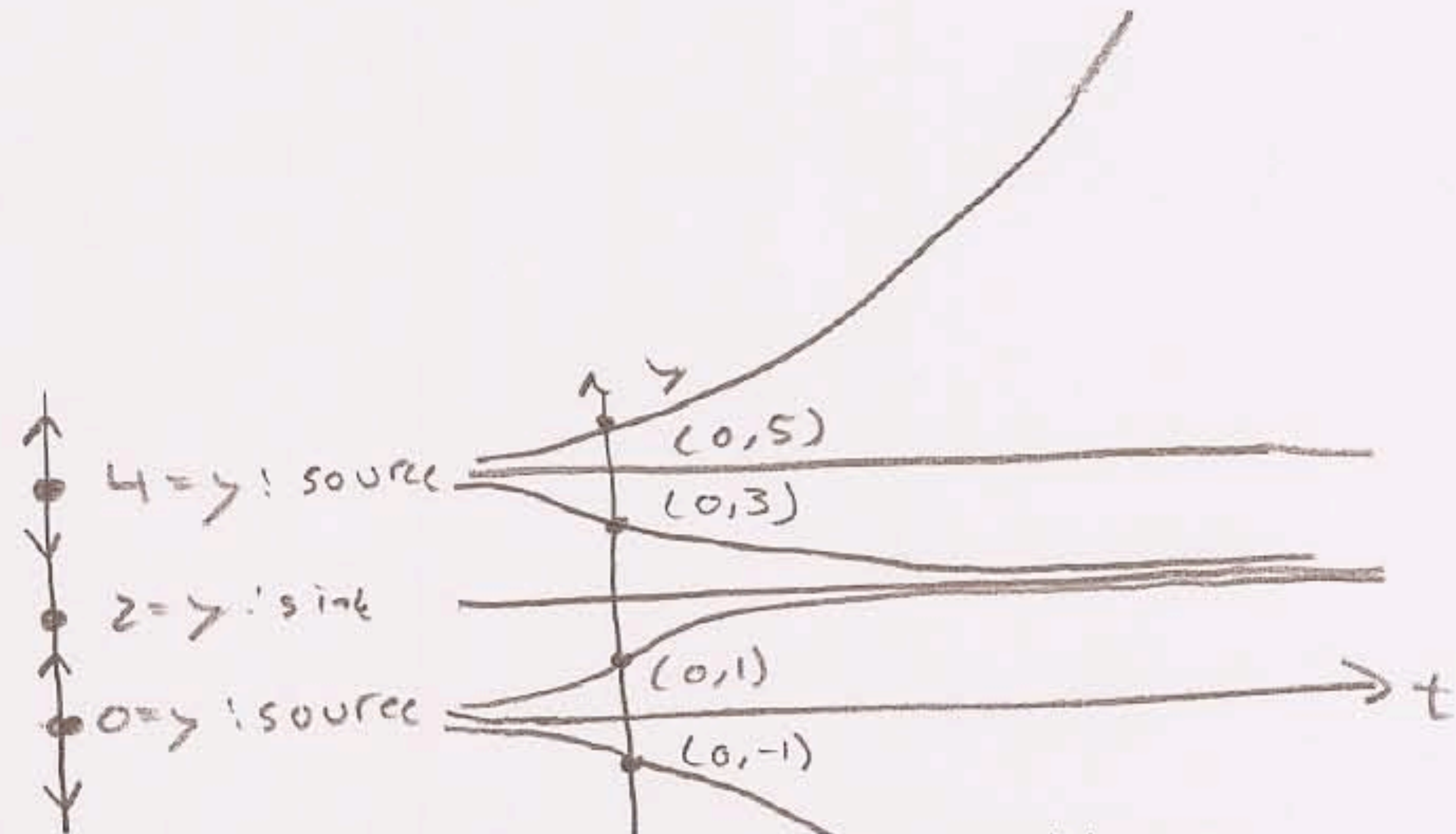
For full credit, you must show all work and box answers.

1. Given $\frac{dy}{dt} = y(y-2)(y-4)$,

   (a) Sketch the phase line and classify all equilibrium points.

   EP: $y(y-2)(y-4)=0$

   $\boxed{y=0,\ y=2,\ y=4}$



   (b) Next to your phase line, sketch the graphs of solutions satisfying the initial conditions $y(0) = -1$, $y(0) = 1$, $y(0) = 3$, and $y(0) = 5$. Put your graphs on one pair of axes.

   (c) Describe the long-term behavior of the solution that satisfies the initial condition $y(0) = 1$.

   $\boxed{y \to 2 \text{ as } t \to \infty,\ y \to 0 \text{ as } t \to -\infty}$

2. Given $\frac{dy}{dt} = \sin(y^2)$,

   (a) Sketch the phase line and classify all equilibrium points. (You cannot sketch the entire phase line in this case, but show at least 5 equilibrium points, including both positive and negative y-values.)

   EP: $\sin(y^2)=0$

   $y^2 = 0, \pi, 2\pi, \dots$

   $\boxed{\begin{array}{l} y = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \dots \\ y = \pm\sqrt{k\pi},\ k=0,1,2,\dots \end{array}}$

   Linearization Thm

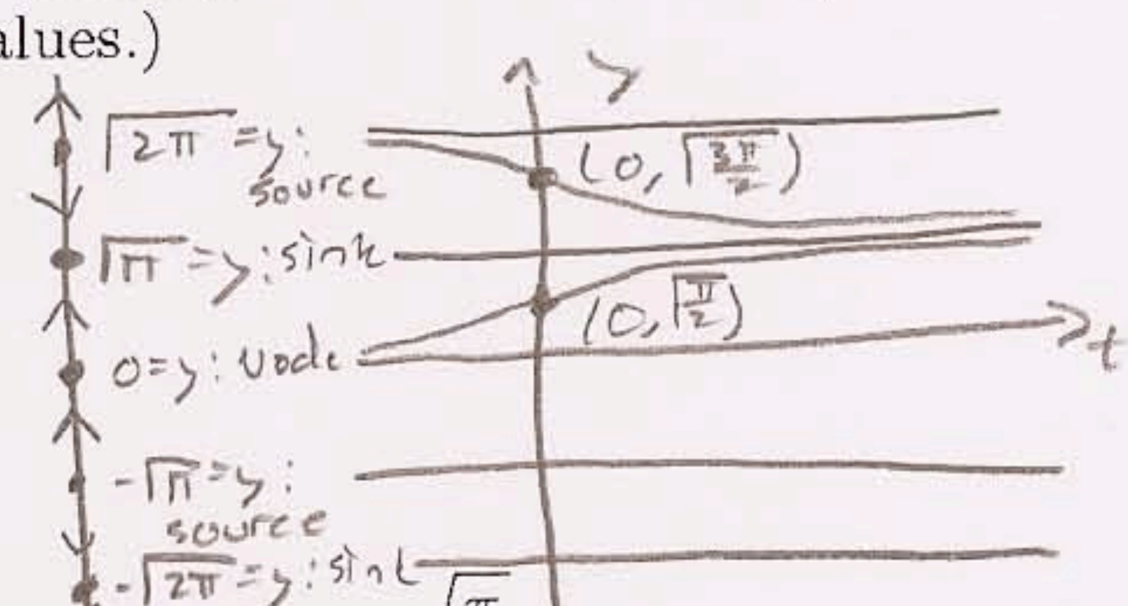
   $f(y) = \sin(y^2)$

   $f'(y) = 2y\cos(y^2)$

   $f'(0) = 0$: ?

   $f'(\sqrt{\pi}) < 0$: sink, $f'(-\sqrt{\pi}) > 0$: source

   $f'(\sqrt{2\pi}) > 0$: source, $f'(-\sqrt{2\pi}) < 0$: sink



   (b) Next to your phase line, sketch the graphs of solutions satisfying the initial conditions $y(0) = \sqrt{\frac{\pi}{2}}$, and $y(0) = \sqrt{\frac{3\pi}{2}}$. Put your graphs on one pair of axes.

   (c) Describe the long-term behavior of the solution that satisfies the initial condition $y(0) = \sqrt{\frac{\pi}{2}}$.

   $\boxed{y \to \sqrt{\pi} \text{ as } t \to \infty,\ y \to 0 \text{ as } t \to -\infty}$

3. For the one-parameter family $\frac{dy}{dt} = y^2 - ay + 4$, $y \in \mathbb{R}$

   (a) Find the bifurcation value(s).

   $f_a(y) = y^2 - ay + 4$

   $f_a'(y) = 2y - a = 0$

   $y = \frac{a}{2}$

   $f_a\left(\frac{a}{2}\right) = \frac{a^2}{4} - \frac{a^2}{2} + 4 = 0$, $\frac{a^2}{4} = 4$, $a^2 = 16$, $\boxed{a = \pm 4}$

   EP: $\frac{dy}{dt} = y^2 - ay + 4 = 0$

   $y = \frac{a \pm \sqrt{a^2-16}}{2}$

   (See attached for classification work.)

   (b) For each bifurcation value, draw phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation value. Make sure to label your graph and classify any equilibrium points.