Tilted window: ray propagation

- Calculate phase shift caused by the insertion of the window into an interferometer.
- Ray optics:
  - Add up optical path for each segment
  - Subtract optical path w/o window

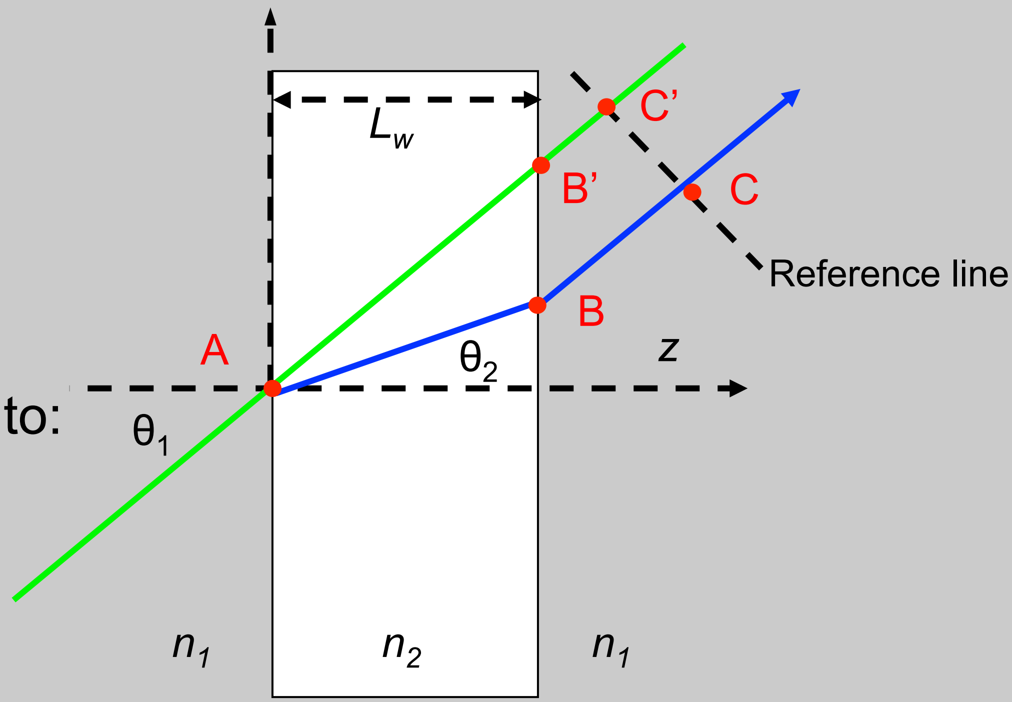
$$\Delta d = nL_{AB} + L_{BC} - L_{AB'} - L_{B'C'}$$

$$L_{AB} = \frac{L_w}{\cos\theta_2} \qquad L_{AB'} = \frac{L_w}{\cos\theta_1}$$

$$L_{BC} = L_{B'C'} + L_{BB'}\sin\theta_1$$

  - Use Snell's Law to reduce to:

$$\Delta d = n\,L_w\cos\theta_2 - L_w\cos\theta_1$$



Tilted window: wave propagation

- Write expression for tilted plane wave

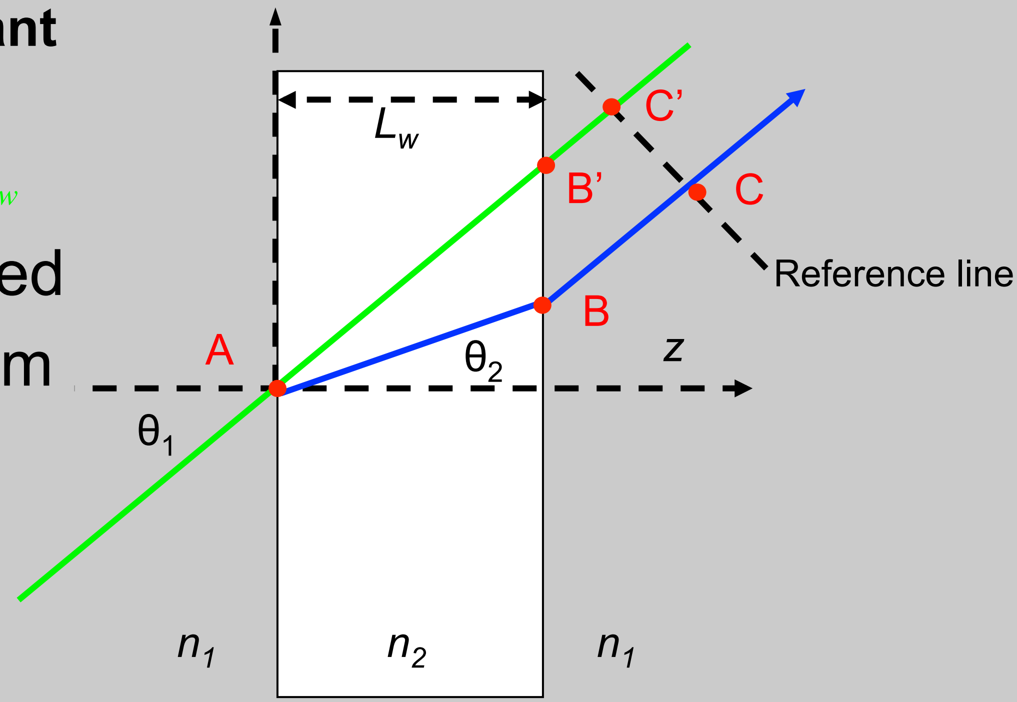
$$E(x,z) = E_0 \exp\left[i\left(k_x x + k_z z\right)\right] = E_0 \exp\left[i\frac{\omega}{c} n\left(x\sin\theta_2 + z\cos\theta_2\right)\right]$$

- Snell's Law: phase across surfaces is conserved

$$k_x x = \frac{\omega}{c} n \sin\theta$$ **is constant**

$$\Delta\phi = \left(k_2 \cos\theta_2\right) L_w - \left(k_1 \cos\theta_1\right) L_w$$

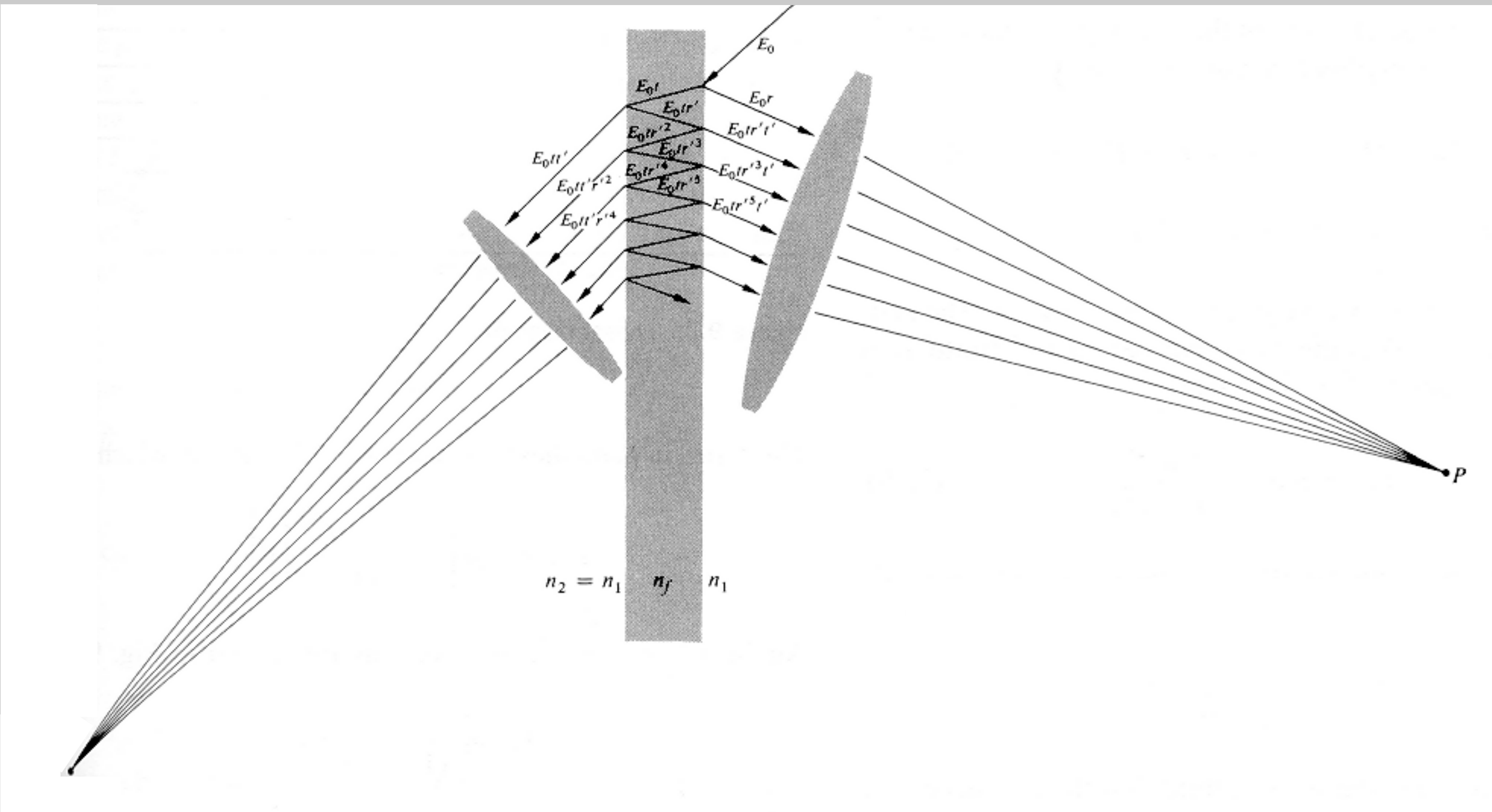
- This approach can be used to calculate phase of prism pairs and grating pairs



Multiple-beam interference: The Fabry-Perot Interferometer or Etalon

A Fabry-Perot interferometer is a pair of **parallel** surfaces that reflect beams back and forth. An etalon is a type of Fabry-Perot etalon, and is a piece of glass with parallel sides.

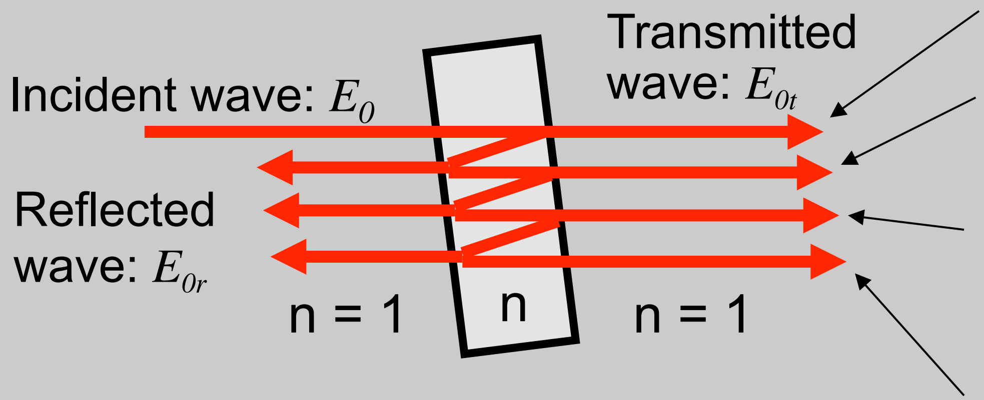
The transmitted wave is an infinite series of multiply reflected beams.



Linear systems approach to the FP

- As with any linear device, we can represent its action in either the temporal or frequency domain
  - Frequency domain: $H(\omega)$ = transfer function
  - Time domain: $h(t)$ = impulse response
- First we will start with the conventional approach:
  - Incident monochromatic plane wave
  - Calculate transmitted amplitude and phase for $H(\omega)$
- Then we should be able to calculate impulse response:
  - $h(t) = FT^{-1}\{ H(\omega) \}$

Calculation of the FP frequency response



$$t t' e^{i\delta/2} E_0$$

$$t t' e^{i\delta/2} \left(r'\right)^2 e^{i\delta} E_0$$

$$t t' e^{i\delta/2} \left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0$$

$$t t' e^{i\delta/2} \left(\left(r'\right)^2 e^{i\delta}\right)^3 E_0$$

r, t **= reflection, transmission coefficients from air to glass**
r', t' **= " " " from glass to air**

δ = round-trip phase delay inside medium = $k_0(2\ n\ L \cos\theta_t)$

$$\delta(\omega) = \omega \frac{2nL}{c} \cos\theta_t(\omega) \approx \omega \frac{2L}{c} \cos\theta_i \quad \text{for n=1}$$

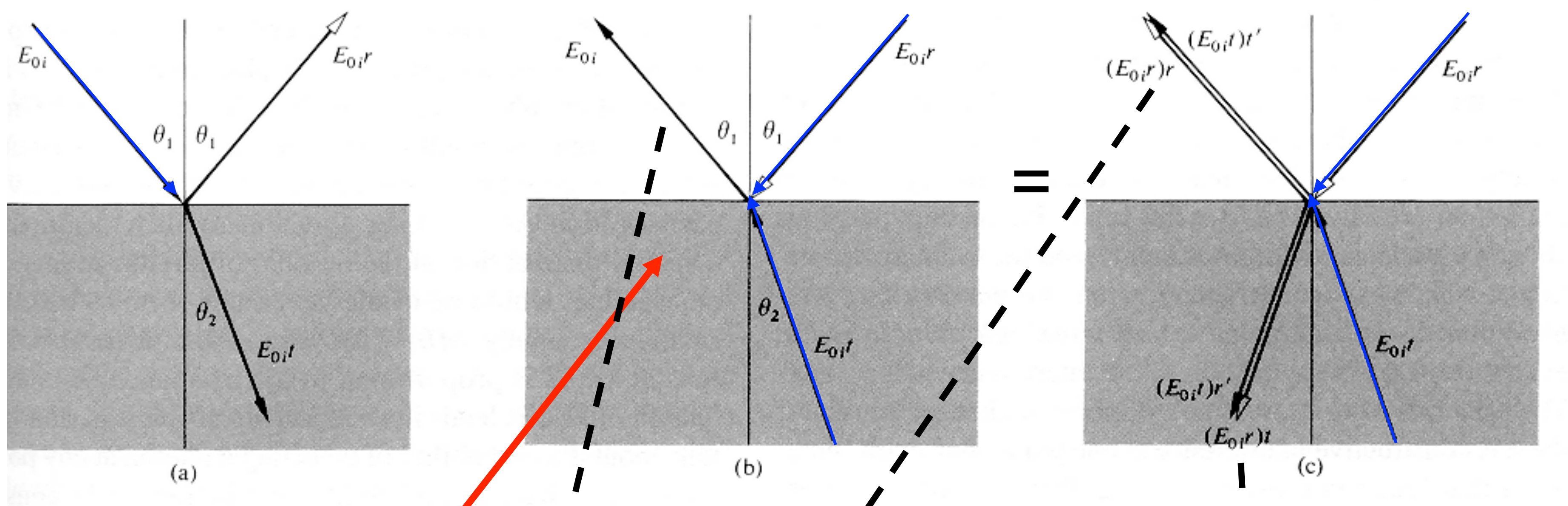
Transmitted wave:

$$E_{0t} = tt' e^{i\delta/2} E_0 \left(1 + \left(r'\right)^2 e^{i\delta} + \left(\left(r'\right)^2 e^{i\delta}\right)^2 + \left(\left(r'\right)^2 e^{i\delta}\right)^3 + \ldots\right)$$

Reflected wave:

$$E_{0r} = rE_0 + tt'r'e^{i\delta} E_0 + tt'r'\left(\left(r'\right)^2 e^{i\delta}\right)^2 E_0 + \ldots$$

Stokes relations for reflection and transmission



“Time reversal:”
Same amplitudes, reversed propagation direction

$$E_{oi} = (E_{oi}r)r + (E_{oi}t)t'$$
$$\therefore tt' = 1 - r^2$$

$$(E_{oi}t)r' + (E_{oi}r)t = 0$$
$$\therefore r' = -r$$

Notes:

- **relations apply to angles connected by Snell’s Law**
- **true for any polarization, but not TIR**
- **convention for which interface experiences a sign change can vary**

Fabry-Perot transfer function

Stokes' relations $r' = -r$, $r'^2 = r^2$, $tt' = 1 - r^2$

The transmitted wave field is:

$$E_{0t} = tt' e^{i\delta/2} E_0 \left(1 + (r')^2 e^{i\delta} + \left((r')^2 e^{i\delta} \right)^2 + \left((r')^2 e^{i\delta} \right)^3 + \ldots \right)$$

$$= tt' e^{i\delta/2} E_0 \left(1 + r^2 e^{i\delta} + \left(r^2 e^{i\delta} \right)^2 + \left(r^2 e^{i\delta} \right)^3 + \ldots \right)$$

Perform sum over infinite series:

Let $x = r^2 e^{i\delta}$

$$E_{0t} = tt' e^{i\delta/2} E_0 \sum_{m=0}^{\infty} \left(r^2 e^{i\delta} \right)^m = \left(1 - r^2 \right) e^{i\delta/2} E_0 \sum_{m=0}^{\infty} x^m$$

For |x|<1: $\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots = (1 - x)^{-1}$

$$\Rightarrow E_{0t} = E_0 \frac{\left(1 - r^2\right) e^{i\delta/2}}{1 - r^2 e^{i\delta}} = E_0(\omega) \frac{\left(1 - r^2\right) \exp\left[i\omega \frac{L}{c} \cos\theta_i \right]}{1 - r^2 \exp\left[i\omega \frac{2L}{c} \cos\theta_i \right]} \equiv E_0(\omega) H(\omega)$$

FP impulse response

- Simple case: n = 1, normal incidence

$$H(\omega)=\frac{\left(1-r^2\right)\exp\left[i\omega\frac{L}{c}\cos\theta_i\right]}{1-r^2\exp\left[i\omega\frac{2L}{c}\cos\theta_i\right]}\rightarrow\frac{1-r^2}{1-r^2\,\mathrm{e}^{i\omega T_{RT}}}$$

Dropping common term:

$$e^{i\omega T_{RT}/2}$$

$$T_{RT}=2L/c$$

- FT^{-1} to get impulse response

$$H(\omega)=\frac{1-r^2}{1-r^2\,\mathrm{e}^{i\omega T_{RT}}}=\left(1-r^2\right)\sum_{m=0}^{\infty}\left(r^2e^{i\delta}\right)^m$$

$$h(t)=FT^{-1}\left\{H(\omega)\right\}=\left(1-r^2\right)\sum_{m=0}^{\infty}r^{2m}FT^{-1}\left\{e^{im\omega T_{RT}}\right\}$$

$$h(t)=\left(1-r^2\right)\sum_{m=0}^{\infty}r^{2m}\delta\left(t-m\omega T_{RT}\right)$$

High reflectivity limit

- When the reflectivity is high, very little is transmitted through the output on each reflection

$r^2 = R$ Power reflectivity

$R^m = (1-T)^m \approx (1-mT) \approx e^{-2m\gamma}$

Logarithmic cavity loss (single pass)

$\gamma = -\ln T = -\ln(1-R)$

- We can represent this as a time dependent loss function:

$L(t) = e^{-t/\tau_c}$ Cavity lifetime: $\tau_c = T_{RT} / 2\gamma$

$$h(t) = (1-r^2)\sum_{m=0}^{\infty} r^{2m}\delta(t - m\omega T_{RT}) \rightarrow (1-r^2)\Theta(t)e^{-t/\tau_c}\,\mathrm{comb}(t / T_{RT})$$

Another calculation of the transfer function

- With this low cavity loss representation of the impulse response, FT to get to H(ω)

$$h(t) = (1 - r^2)\Theta(t)e^{-t/\tau_c}\,\mathrm{comb}(t / T_{RT}) = (1 - r^2) f(t) g(t)$$

$$H(\omega) = FT\{h(t)\} = (1 - r^2)\frac{1}{2\pi}F(\omega)\otimes G(\omega)$$

$$F(\omega) = \int_0^\infty e^{-t/\tau_c} e^{i\omega t}\, dt = \left.\frac{e^{-t/\tau_c + i\omega t}}{-1/\tau_c + i\omega}\right|_0^\infty = \frac{1}{1/\tau_c - i\omega}$$

Complex Lorentzian

$$G(\omega) = FT\{\mathrm{comb}(t / T_{RT})\} = \frac{2\pi}{T_{RT}}\mathrm{comb}\left(\frac{\omega}{2\pi / T_{RT}}\right)$$

Fabry-Perot power transmission

$$E_{0t} = E_0 \frac{1-r^2}{1-r^2 e^{i\delta}} e^{i\delta/2}$$

Power transmittance: $T \equiv \left| \frac{E_{0t}}{E_0} \right|^2 = \left| \frac{\left(1-r^2\right) e^{i\delta/2}}{1-r^2 e^{i\delta}} \right|^2 = \frac{\left(1-r^2\right)^2}{(1-r^2 e^{+i\delta})(1-r^2 e^{-i\delta})}$

$$= \left[\frac{(1-r^2)^2}{\{1+r^4-2r^2\cos(\delta)\}} \right] = \left[\frac{(1-r^2)^2}{\{1+r^4-2r^2[1-2\sin^2(\delta/2)]\}} \right] = \left[\frac{(1-r^2)^2}{\{1-2r^2+r^4+4r^2\sin^2(\delta/2)]\}} \right]$$

Dividing numerator and denominator by $(1-r^2)^2$

$$T = \frac{1}{1+F\sin^2\left(\delta/2\right)}$$

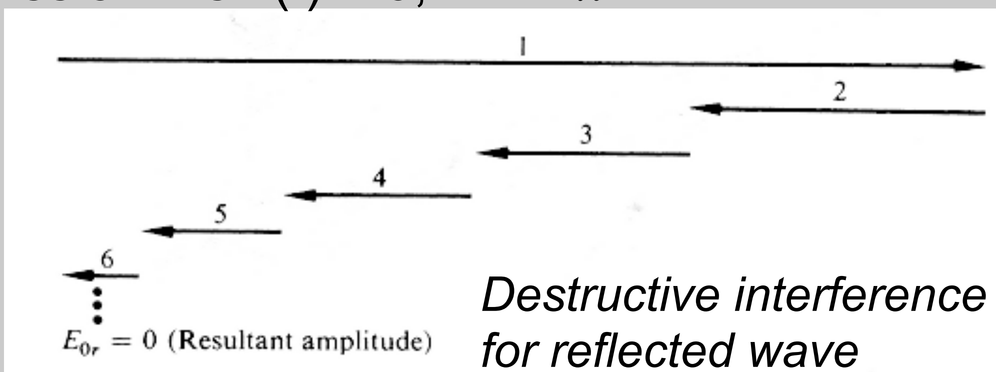
where: $F = \left[\frac{2r}{1-r^2} \right]^2$

Multiple-beam interference: simple limits

Reflected waves

$$T = \frac{1}{1 + F \sin^2\left(\delta / 2\right)}$$

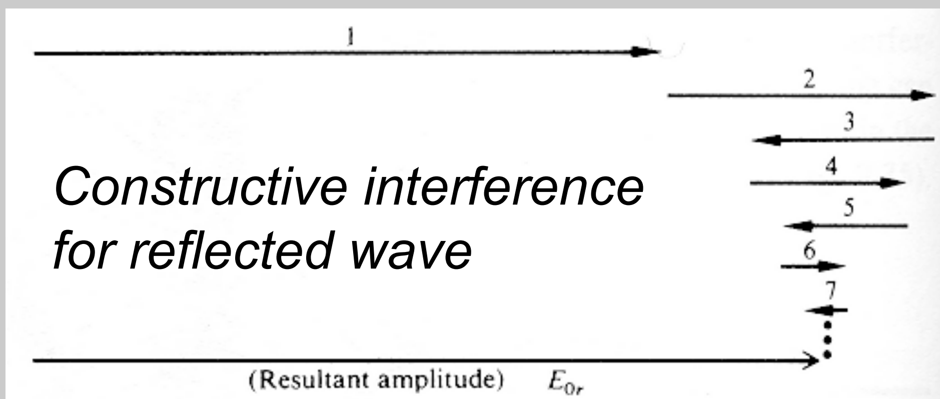
Full transmission: sin( ) = 0, d = 2 π m



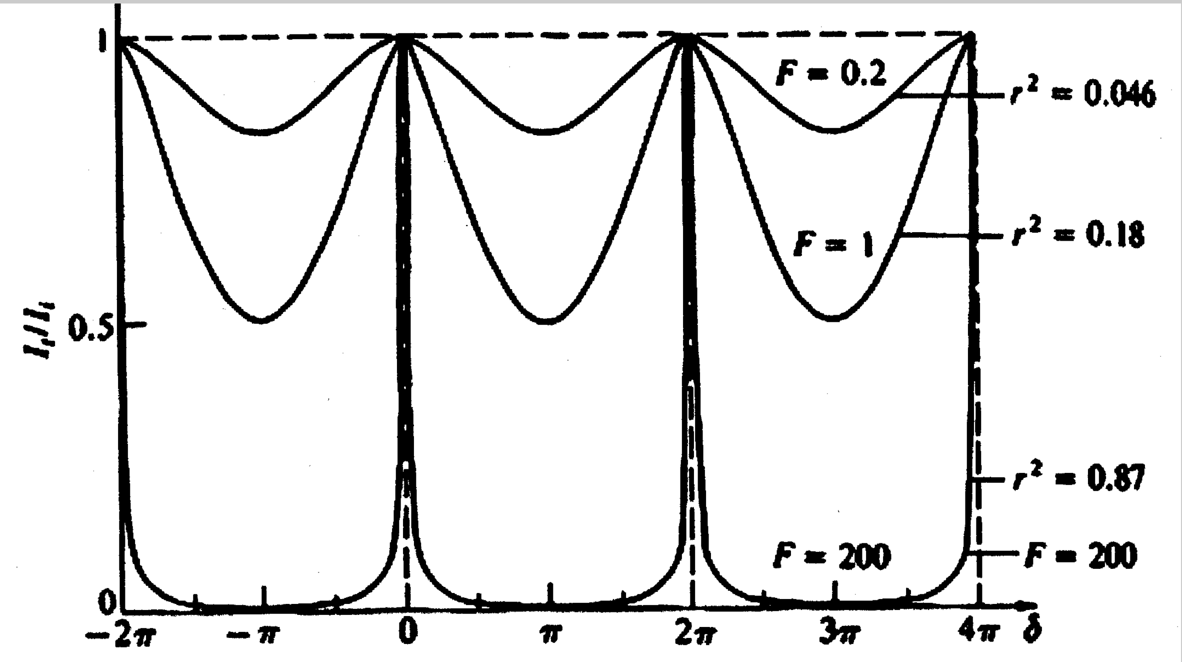
1st reflection

internal reflections

Minimum transmission: sin( ) = 1, d = 2 π (m+1/2)



Etalon transmittance vs. thickness, wavelength, or angle



$$T = \frac{1}{1 + F\sin^2(\delta/2)}$$

Transmission max:
sin( ) = 0, d = 2 π m

$$\delta = \frac{\omega}{c} 2nL\cos[\theta_t]$$
$$= 2\pi m$$

At normal incidence:

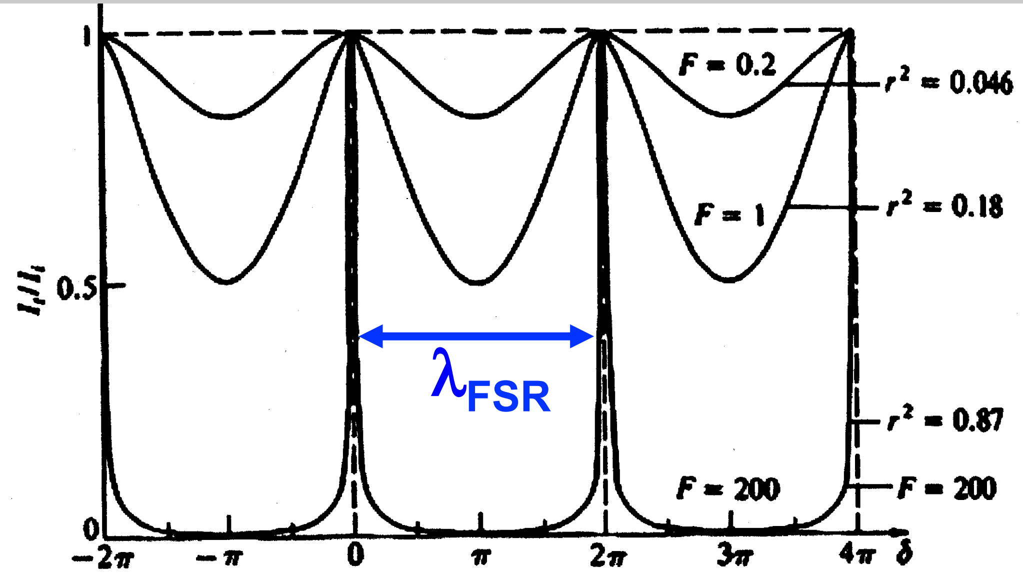
$$\lambda_m = \frac{2nL}{m} \quad \text{or} \quad nL = m\frac{\lambda_m}{2}$$

- The transmittance varies significantly with thickness or wavelength.
- We can also vary the incidence angle, which also affects δ.
- As the reflectance of each surface ($R=r^2$) approaches 1, the widths of the high-transmission regions become very narrow.

The Etalon Free Spectral Range

The Free Spectral Range is the wavelength range between transmission maxima.

λ_{FSR} = Free Spectral Range



For neighboring orders:

$$\frac{4\pi nL}{\lambda_1} - \frac{4\pi nL}{\lambda_2} = 2\pi \Rightarrow \frac{1}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1}{2nL} = \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}$$

$$\lambda_2 - \lambda_1 = \lambda_{FSR}$$

$$\lambda_2 \lambda_1 \approx \lambda^2$$

$$\lambda_{FSR} \approx \frac{\lambda^2}{2nL}$$

$$\frac{\lambda_{FSR}}{\lambda} = \frac{\lambda}{2nL} = \frac{\nu_{FSR}}{\nu}$$

$$\nu_{FSR} \approx \frac{c}{2nL}$$

1/(round trip time)

Etalon Linewidth

The Linewidth δ_{LW} is a transmittance peak's full-width-half-max (FWHM).

$$T = \frac{1}{1 + F\sin^2\left(\delta / 2\right)}$$

A maximum is where $\delta / 2 \approx m\pi + \delta' / 2$ and $\sin^2\left(\delta / 2\right) \approx \delta' / 2$

Under these conditions (near resonance),

$$T = \frac{1}{1 + F\delta'^2 / 4}$$

This is a Lorentzian profile, with FWHM at:

$$\frac{F}{4}\left(\frac{\delta_{LW}}{2}\right)^2 = 1 \quad \Rightarrow \quad \delta_{LW} \approx 4 / \sqrt{F}$$

This transmission linewidth corresponds to the minimum resolvable wavelength.

Etalon Finesse

The Finesse, $\Im$, is the ratio of the
Free Spectral Range and the Linewidth:

$$\Im \equiv \frac{\delta_{FSR}}{\delta_{FW}} = \frac{2\pi}{4/\sqrt{F}} = \frac{\pi\sqrt{F}}{2}$$

$\delta = 2\pi$ corresponds to one FSR

Using: $$F = \left[\frac{2r}{1-r^2}\right]^2$$

$$\boxed{\Im = \frac{\pi}{1-r^2}}$$

taking $r \approx 1$

The Finesse is the number of wavelengths the interferometer can resolve.