

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Where appropriate, please enclose your final answers in boxes.

1. (26 points) Consider the differential equation:  $y'' - 8y' + 16y = f(t)$ . Solve under the following conditions:

a.  $f(t) = 0$

$$y'' - 8y' + 16y = 0$$

$$y_h(t) = y_h: y_h'' - 8y_h' + 16y_h = 0, y_h = e^{rt}$$

$$r^2 - 8r + 16 = 0$$

$$(r-4)^2 = 0, r = 4$$

$$y(t) = k_1 e^{4t} + k_2 t e^{4t}$$

b.  $f(t) = 5e^{2t}$

$$y'' - 8y' + 16y = 5e^{2t}$$

$$y_h(t) = k_1 e^{4t} + k_2 t e^{4t} \text{ (See above)}$$

$$y_p: y_p'' - 8y_p' + 16y_p = 5e^{2t}$$

$$y_p = \alpha e^{2t}, y_p' = 2\alpha e^{2t}, y_p'' = 4\alpha e^{2t}$$

$$4\alpha e^{2t} - 16\alpha e^{2t} + 16\alpha e^{2t} = 5e^{2t}$$

$$4\alpha e^{2t} = 5e^{2t}$$

$$4\alpha = 5, \alpha = \frac{5}{4}, y_p = \frac{5}{4} e^{2t}$$

c. Now assume  $y(0) = 0, y'(0) = 6, f(t) = \delta_3(t)$

$$y'' - 8y' + 16y = \delta_3(t)$$

$$\mathcal{L}[y''] - 8\mathcal{L}[y'] + 16\mathcal{L}[y] = \mathcal{L}[\delta_3(t)]$$

$$s^2 \mathcal{L}[y] - (0)s - 6 - 8(s \mathcal{L}[y] - 0) + 16 \mathcal{L}[y] = e^{-3s}$$

$$(s^2 - 8s + 16) \mathcal{L}[y] = e^{-3s} + 6$$

$$\mathcal{L}[y] = e^{-3s} \left( \frac{1}{s^2 - 8s + 16} \right) + \frac{6}{s^2 - 8s + 16}$$

$$\mathcal{L}[y] = e^{-3s} \left( \frac{1}{(s-4)^2} \right) + \frac{6}{(s-4)^2}$$

$$\mathcal{L}[y] = e^{-3s} \left( \frac{1}{(s-4)^2} \right) + 6 \left( \frac{1}{(s-4)^2} \right)$$

$$y = \mathcal{L}^{-1}[\mathcal{L}[y]]$$

$$y(t) = u_3(t) e^{4(t-3)} (t-3) + 6e^{4t} t$$

2. (26 points) Consider a harmonic oscillator governed by the equation:  $y'' + 9y = 10\sin(2t)$ ,  $y(0) = v(0) = 0$ . Find position at any time using the following methods:

a. The Method of Undetermined Coefficients

$$y'' + 9y = 10\sin(2t)$$

$$y_h: y_h'' + 9y_h = 0, \quad y_h = e^{rt}$$

$$r^2 + 9 = 0$$

$$r^2 = -9, \quad r = \pm 3i$$

$$e^{rt} = e^{3it} = \cos(3t) + i\sin(3t)$$

$$y_h(t) = k_1 \cos(3t) + k_2 \sin(3t)$$

$$y_c: y_c'' + 9y_c = 10e^{2it}$$

$$= 10 \cos(2t) + i(10) \sin(2t)$$

$$y_c = \alpha e^{2it}$$

$$y_c' = 2i\alpha e^{2it}, \quad y_c'' = -4\alpha e^{2it}$$

$$-4\alpha e^{2it} + 9\alpha e^{2it} = 10e^{2it}$$

$$5\alpha = 10$$

$$\alpha = 2$$

b. Laplace transforms

$$y'' + 9y = 10\sin(2t)$$

$$\mathcal{L}[y''] + 9\mathcal{L}[y] = 10\mathcal{L}[\sin(2t)]$$

$$s^2 \mathcal{L}[y] - 0 - 0 + 9\mathcal{L}[y] = 10 \left( \frac{2}{s^2 + 2^2} \right)$$

$$(s^2 + 9)\mathcal{L}[y] = \frac{20}{(s^2 + 2^2)}$$

$$\mathcal{L}[y] = \frac{20}{(s^2 + 4)(s^2 + 9)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 9}$$

$$\mathcal{L}[y] = \frac{4}{s^2 + 4} - \frac{4}{s^2 + 9} = 2 \left( \frac{2}{s^2 + 2^2} \right) - \frac{4}{3} \left( \frac{3}{s^2 + 3^2} \right)$$

$$y = \mathcal{L}^{-1}[\mathcal{L}[y]]$$

$$y(t) = 2 \sin(2t) - \frac{4}{3} \sin(3t)$$

$$y_c = 2e^{2it}$$

$$= 2\cos(2t) + i(\underbrace{2\sin(2t)}_{\text{Imag.}})$$

$$y_p = 2\sin(2t)$$

$$y(t) = k_1 \cos(3t) + k_2 \sin(3t)$$

$$+ 2\sin(2t)$$

$$y'(t) = -3k_1 \sin(3t) + 3k_2 \cos(3t)$$

$$+ 4\cos(2t)$$

$$y(0) = k_1 = 0$$

$$y'(0) = 3k_2 + 4 = 0, \quad k_2 = -\frac{4}{3}$$

$$y(t) = -\frac{4}{3} \sin(3t) + 2\sin(2t)$$

### Partial Fractions

$$(As + B)(s^2 + 9) + (Cs + D)(s^2 + 4) = 20$$

~~$$As^3 + 9As^2 + Bs^2 + 9B + Cs^3 + 4Cs^2 + Ds^2 + 4Ds = 20$$~~

$$+ Ds^2 + 4Ds = 20$$

$$\begin{array}{lll} A + C = 0 & B + D = 0 & 9A + 4C = 0 \\ C = -A & B = -D & 5A = 0 \\ C = 0 & B = 4 & A = 0 \\ & & D = -4 \end{array}$$

2. (26 points) Consider a harmonic oscillator governed by the equation:  $y'' + 9y = 10\sin(2t)$ ,  $y(0) = v(0) = 0$ . Find position at any time using the following methods:

a. The Method of Undetermined Coefficients

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

$$y_h = C_1 \cos(3t) + C_2 \sin(3t)$$

$$\text{Guess: } y_p = A\sin(2t) + B\cos(2t)$$

$$\Rightarrow y_p' = 2A\cos(2t) - 2B\sin(2t)$$

$$y_p'' = -4A\sin(2t) - 4B\cos(2t)$$

$$-4A\sin(2t) - 4B\cos(2t) + 9A\sin(2t) + 9B\cos(2t) = 10\sin(2t)$$

$$\Rightarrow B=0, 5A=10 \Rightarrow A=2$$

$$y = C_1 \cos(3t) + C_2 \sin(3t) + 2\sin(2t)$$

$$y' = -3C_1 \sin(3t) + 3C_2 \cos(3t) + 4\cos(2t)$$

$$y(0) = C_1 = 0, y'(0) = 3C_2 + 4 = 0 \Rightarrow C_2 = -\frac{4}{3}$$

$$y = -\frac{4}{3} \sin(3t) + 2\sin(2t)$$

b. Laplace transforms

3. (10 points) Consider the problem  $y'' + y = \cos(\omega t)$

a. Find the homogeneous part of the solution.

$$y_h: y_h'' + y_h = 0, y_h = e^{rt}$$

$$r^2 + 1 = 0, r^2 = -1$$

$$r = \pm i$$

$$e^{rt} = e^{it} = \cos t + i \sin t$$

$$y_h(t) = k_1 \cos t + k_2 \sin t$$

$$\omega_0 = 1$$

- b. For  $\omega = 1$ , what phenomenon is occurring in this system?

$$\omega_0 = \omega = 1$$

Resonance

- c. For  $\omega = 0.6$ , what phenomenon is occurring in this system?

Beats or Beating

- d. Match the following values of omega to the corresponding graph:

i.  $\omega = 0.7$

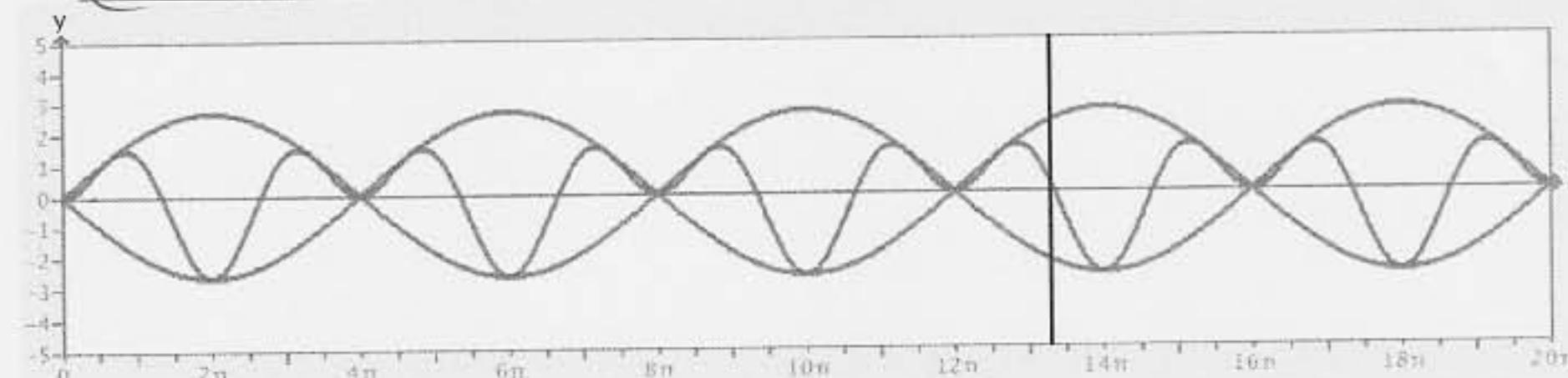
ii.  $\omega = 1$

iii.  $\omega = 0.5$

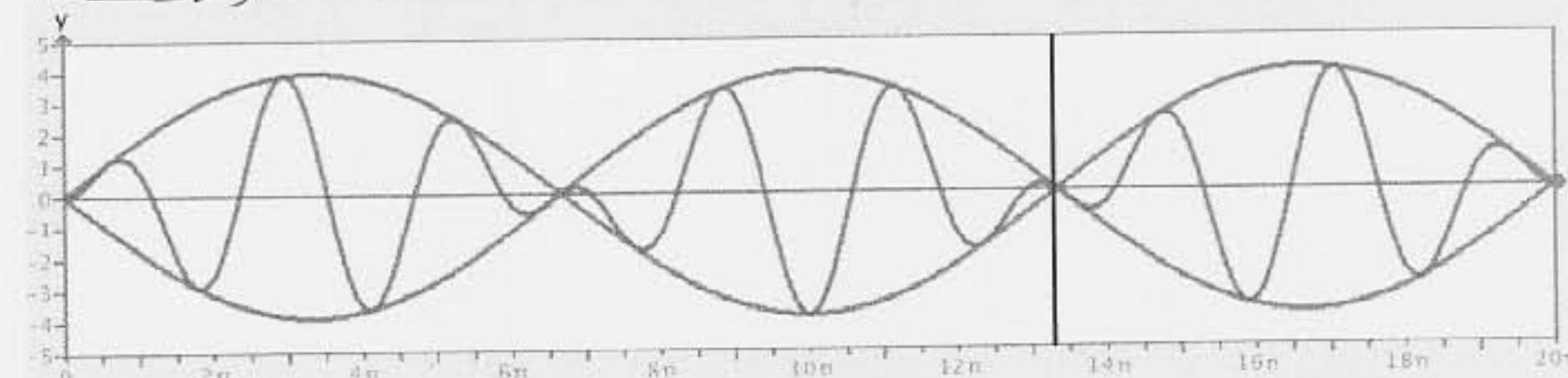
A. (ii)



B. (iii)



C. (i)



4. (16 points) Using the power series method on the ODE  $y' - 4y = 0$

a. Find the recurrence relation.

$$y = \sum_{n=0}^{\infty} a_n t^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\sum_{n=1}^{\infty} n a_n t^{n-1} - 4 \sum_{n=0}^{\infty} a_n t^n = 0$$
$$\sum_{n=1}^{\infty} n a_n t^{n-1} - \sum_{n=1}^{\infty} 4 a_n t^n = 0$$
$$n a_n - 4 a_{n-1} = 0$$
$$\boxed{a_n = 4 \frac{a_{n-1}}{n}, n \geq 1}$$

b. Find the power series solution and then, using a known Taylor Series, write your answer in the form  $y = f(t)$ .

$$a_0$$

$$a_1 = 4a_0$$

$$a_2 = \frac{4}{2} a_1 = \frac{4^2}{2} a_0$$

$$a_3 = \frac{4}{3} a_2 = \frac{4^3}{3!} a_0$$

$$\vdots$$
$$a_n = \frac{4^n}{n!} a_0$$

$$y = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} \frac{4^n}{n!} a_0 t^n = a_0 \sum_{n=0}^{\infty} \frac{(4t)^n}{n!} = \boxed{a_0 e^{4t}}$$

5. (12 points) Using the table provided at the beginning of the exam, calculate the inverse Laplace transform of the following functions:

$$a. F(s) = \frac{s+6}{s^2 + 4s + 13} = \frac{s+6}{s^2 + 4s + 4 + 13 - 4} = \frac{s+2+4}{(s+2)^2 + 9}$$

$$= \frac{s+2}{(s+2)^2 + 9} + \frac{4}{3} \frac{3}{(s+2)^2 + 9}$$

$$\boxed{f(t) = e^{-2t} \cos(3t) + \frac{4}{3} e^{-2t} \sin(3t)}$$

$$b. G(s) = \frac{3e^{-s}}{s^2 + 4} = \frac{3}{2} e^{-s} \frac{2}{s^2 + 4}$$

$\Downarrow \sin(2t)$

$$\boxed{g(t) = \frac{3}{2} u(t-1) \sin(2(t-1))}$$

6. (10 points) Solve the following initial value problem:  $2y' + 4y = u_3(t), y(0) = 12$

$$2(sY(s) - 12) + 4Y(s) = \frac{e^{-3s}}{s}$$

$$(2s+4)Y(s) = \frac{e^{-3s}}{s} + 24$$

$$Y(s) = e^{-3s} \frac{1}{s(2s+4)} + \frac{12}{s+2}$$

$$\frac{1}{s(2s+4)} = \frac{A}{s} + \frac{B}{2s+4}$$

$$1 = 2As + 4A + Bs$$

$$s: 2A + B = 0$$

$$\# : 4A = 1 \Rightarrow A = \frac{1}{4}, B = -\frac{1}{2}$$

$$Y(s) = e^{-3s} \left[ \frac{1}{4} \frac{1}{s} - \frac{1}{4} \frac{1}{s+2} \right] + \frac{12}{s+2}$$

$$\boxed{y(t) = u(t-3) \left[ \frac{1}{4} - \frac{1}{4} e^{-2(t-3)} \right] + 12 e^{-2t}}$$