Equation of Motion

We have a 3D dipole with dipole moment $\boldsymbol{\mu} = \mu\,\hat{\mathbf{z}}$. It moves on the axis of a circular loop of radius *a*, resistance *R*, inductance *L*, with inductive time constant *L/R*. It moves downward under the influence of gravity. We constrain the motion to be along the *z*-axis, and the magnetic dipole moment to be parallel to that axis. The equation of motion is

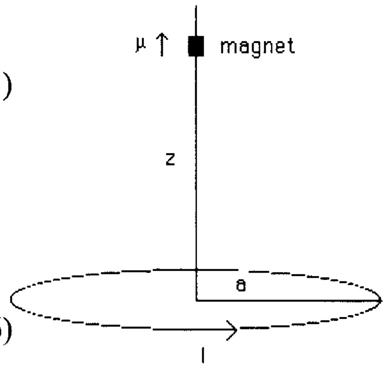
$$m\frac{d^2z}{dt^2} = -mg + \mu\frac{dB_z}{dz} \tag{3}$$

where B_Z is the field due the current *I* in the ring (taken to be positive in the direction show on the sketch). The expression for B_Z is

$$B_z = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}} \tag{4}$$

so that equation (3) is

$$m\frac{d^2z}{dt^2} = -mg - \frac{3\mu\mu_o I a^2}{2}\frac{z}{(a^2+z^2)^{5/2}} \tag{5}$$



An Equation for I from Faraday's Law

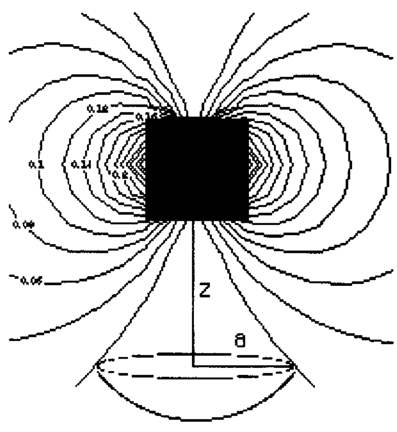
Faraday's Law is

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int\left[\mathbf{B}_{dipole} + \mathbf{B}_{ring}\right]\cdot d\mathbf{A} = -\frac{d}{dt}\int \mathbf{B}_{dipole}\cdot d\mathbf{A} - L\frac{dI}{dt} \tag{6}$$

where L is the inductance of the ring. If $\mathbf{E} = \rho\mathbf{J}$, where ρ is the resistivity of the ring and **J** is the current density, then $\oint \mathbf{E}\cdot d\mathbf{l} = \oint \rho\,\mathbf{J}\cdot d\mathbf{l} = I\oint \rho\, dl/A = IR$, with $R = \oint \rho\, dl/A$ where *R* is the resistance of the ring. So we have

$$IR = -L\frac{dI}{dt} - \frac{d}{dt}\int \mathbf{B}_{dipole}\cdot d\mathbf{A} \tag{7}$$

We now need to determine the magnetic flux through the ring due to the dipole field. To do this we calculate the flux through a spherical cap of radius $\sqrt{a^2+z^2}$ with an opening angle given θ given by $\sin\theta = a/\sqrt{a^2+z^2}$ (this is the same as the flux through the ring because $\nabla\cdot\mathbf{B} = 0$). The flux through a spherical cap only involves the radial component of the dipole field, given by



$$B_r = \frac{\mu_o}{2\pi}\frac{\mu\cos\theta}{r^3}$$

$$\int \mathbf{B}_{dipole}\cdot d\mathbf{A} = \int \frac{\mu_o}{2\pi}\frac{\mu\cos\theta}{r^3}2\pi r^2 \sin\theta\, d\theta$$

$$\int \mathbf{B}_{dipole}\cdot d\mathbf{A} = -\frac{\mu_o\mu}{r}\int \cos\theta\, d\cos\theta = \frac{\mu_o\mu}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}$$

Using this expression for the flux in (7), and assuming that $\mu = M_O$ is constant in time, yields

$$IR = -L\frac{dI}{dt} + \frac{3\mu_o a^2 M_o}{2}\frac{z}{\left(a^2+z^2\right)^{5/2}}\frac{dz}{dt} \tag{8}$$

Equations (5) and (8) are our coupled ordinary differential equations which determine the dynamics of the situation when the magnet is falling toward the ring under the influence of gravity.