

are given by,

$$(26) \quad a_i = \frac{2}{R^2 J_{n+1}^2(k_{n,i}R)} \int_0^R x J_n(k_{n,i}R) f(x) dx, \quad i = 1, 2, 3, \dots$$

We will explicitly find these solutions in class but the practical matter is how to use them. If someone provides the functions and orthogonality condition then we can cavalierly expand in this basis.<sup>2</sup> Before we do this we should notice,

- The relation (24) implies that the orthogonality is on the argument not the specific Bessel function in question. That is, we have infinitely many Bessel functions forming a class indexed by the integer  $n$  and the functions in the class are mutually orthogonal. This is seen by the fact that the Kronecker delta function has index  $(m, i)$  and not  $(m, n)$ .

We now proceed, as usual, by considering an appropriately chosen inner-product on (25).<sup>3</sup> Doing so gives,

$$(27) \quad \langle J_n(k_{m,i}x), f(x) \rangle = \left\langle J_i(k_{m,i}x), \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) \right\rangle$$

$$(28) \quad = \sum_{m=1}^{\infty} a_m \langle J_n(k_{m,i}x), J_n(k_{n,m}x) \rangle$$

$$(29) \quad = \sum_{m=1}^{\infty} a_m \frac{\delta_{mi}}{2} [RJ_{n+1}(k_{nm}R)]^2$$

$$(30) \quad = a_i \frac{\delta_{ii}}{2} [RJ_{n+1}(k_{ni}R)]^2,$$

which implies that,

$$(31) \quad a_i = \frac{2}{[RJ_{n+1}(k_{ni}R)]^2} \langle J_n(k_{m,i}x), f(x) \rangle$$

$$(32) \quad = \frac{2}{[RJ_{n+1}(k_{n,i}R)]^2} \int_0^R x f(x) J_n(k_{m,i}x) dx.$$

## 2. POWER-SERIES SOLUTIONS TO ODE'S AND HYPERBOLIC TRIGONOMETRIC FUNCTIONS

Consider the ordinary differential equation:

$$(33) \quad y'' - y = 0$$

2.1. **General Solution - Standard Form.** Show that the solution to (33) is given by  $y(x) = c_1 e^x + c_2 e^{-x}$ .

You could derive this solution from (33) but the beauty of a differential equation is that you can always check.

$$(34) \quad y'' - y = c_1 e^x + (-1)^2 c_2 e^{-x} - c_1 e^x - c_2 e^{-x} = 0.$$

2.2. **General Solution - Nonstandard Form.** Show that  $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$  is a solution to (33) where  $\sinh(x) = \frac{e^x - e^{-x}}{2}$  and  $\cosh(x) = \frac{e^x + e^{-x}}{2}$ .

First we notice the derivative relations on the hyperbolic trigonometric functions,

$$\begin{aligned} y(x) &= b_1 \sinh(x) + b_2 \cosh(x), \\ y'(x) &= b_1 \cosh(x) + b_2 \sinh(x), \\ y''(x) &= b_1 \sinh(x) + b_2 \cosh(x), \end{aligned}$$

which then gives,

$$(35) \quad y'' - y = b_1 \sinh(x) + b_2 \cosh(x) - (b_1 \sinh(x) + b_2 \cosh(x)) = 0,$$

and shows that this is also a solution to (33).

<sup>2</sup>You should remember that you could owe some mathematician money at this point but hey a trade is a trade.

<sup>3</sup>See HW2.5.4 for a reminder.

**2.3. Conversion from Standard to Nonstandard Form.** Show that if  $c_1 = \frac{b_1 + b_2}{2}$  and  $c_2 = \frac{b_1 - b_2}{2}$  then  $y(x) = c_1 e^x + c_2 e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$ .

Direct substitution gives,

$$\begin{aligned} y(x) &= c_1 e^x + c_2 e^{-x}, \\ &= \left(\frac{b_1 + b_2}{2}\right) e^x + \left(\frac{b_1 - b_2}{2}\right) e^{-x}, \\ &= \frac{b_1(e^x + e^{-x})}{2} + \frac{b_2(e^x - e^{-x})}{2}, \\ &= b_1 \cosh(x) + b_2 \sinh(x), \end{aligned}$$

which shows the two solutions are equivalent.

**2.4. Relation to Power-Series.** Assume that  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  to find the general solution of (33) in terms of the hyperbolic sine and cosine functions.<sup>4</sup>

Assume that  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  and find the general solution of (33). First, we have the following derivative relations.

$$(37) \quad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$(38) \quad y'(x) = \sum_{n=0}^{\infty} a_n(n) x^{n-1}$$

$$(39) \quad y''(x) = \sum_{n=0}^{\infty} a_n(n)(n-1) x^{n-2}.$$

Thus the ODE is now,

$$(40) \quad y'' - y = \sum_{n=0}^{\infty} a_n(n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n$$

$$(41) \quad = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=0}^{\infty} a_k x^k$$

$$(42) \quad = \sum_{k=0}^{\infty} [a_{k+2}(k+2)(k+1) - a_k] x^k = 0.$$

The only polynomial, which is zero regardless of its variable is the zero-function. This defines the recurrence relation,

$$(43) \quad [a_{k+2}(k+2)(k+1) - a_k] = 0, \text{ for } k = 1, 2, 3, \dots$$

We now write down  $a_{k+2}$  in terms of  $a_k$ ,

$$(44) \quad a_{k+2} = \frac{a_k}{(k+2)(k+1)}, \text{ for } k = 1, 2, 3, \dots,$$

and try to find a pattern by choosing various  $k$ . Since the recurrence relation relates a coefficient to the coefficient two away it splits the coefficients into even and odd patterns.

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<sup>4</sup>The hyperbolic sine and cosine have the following Taylor's series representations centred about  $x = 0$ ,

$$(36) \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when  $i$  is raised to integer powers it will produce these exact sign alternations. So, if you remember  $e^x = \sum_{n=0}^{\infty} x^n/n!$  and  $i = \sqrt{-1}$  then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

Even Coefficients	Odd Coefficients
$a_2 = \frac{a_0}{2 \cdot 1}$	$a_3 = \frac{a_1}{3 \cdot 2 \cdot 1}$
$a_4 = \frac{a_4}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$	$a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$\vdots$	$\vdots$
$a_{2k} = \frac{a_0}{(2k)!}$	$a_{2k+1} = \frac{a_1}{(2k+1)!}$

The last entries are the generalization of the pattern where the following are noticed:<sup>5</sup>

- (1) The even coefficients all re-curse back to  $a_0$  and the odd coefficients recurse back to  $a_1$ . Nothing more can be said about these two coefficients.
- (2) As the recursion is applied a product forms in the denominator. This product is precisely the factorial of the coefficient's subscript.

Using this information we now have the following,

$$\begin{aligned}
 (45) \quad y(t) &= \sum_{n=0}^{\infty} a_n x^n \\
 (46) \quad &= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\
 (47) \quad &= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\
 (48) \quad &= a_0 \cosh(x) + a_1 \sinh(x),
 \end{aligned}$$

### 3. CONSERVATION LAWS IN ONE-DIMENSION

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$(49) \quad \frac{\partial u}{\partial t} = -\kappa \nabla \phi,$$

which reduces to

$$(50) \quad \frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x}, \quad \kappa \in \mathbb{R}$$

in one-dimension of space.<sup>6</sup> In general, if the function  $u = u(x, t)$  represents the density of a physical quantity then the function  $\phi = \phi(x, t)$  represents its flux. If we assume the  $\phi$  is proportional to the negative gradient of  $u$  then, from (50), we get the one-dimensional heat/diffusion equation.<sup>7</sup>

**3.1. Transport Equation.** Assume that  $\phi$  is proportional to  $u$  to derive, from (50), the convection/transport equation,  $u_t + cu_x = 0$   $c \in \mathbb{R}$ .

<sup>5</sup>A generalized pattern is not always possible to find. However, it should be noted that once the recursion relation is found the job is done. Every coefficient can be known in terms of ones that precede connecting coefficients to  $a_0$  and/or  $a_1$ . These originators are actually the same as the unknown constants  $c_1$  and  $c_2$  found in the homogeneous solution and can be found via initial conditions. Thus, initial conditions will allow the evaluation of the infinite series to arbitrary decimal precision, which is good enough for tunnel work, as they say.

<sup>6</sup>When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity  $u$  could be charge density and  $q$  would be its flux.

<sup>7</sup>AKA Fick's Second Law associated with linear non-steady-state diffusion.