MATH348-Advanced Engineering Mathematics		**Homework: One**

Vector Spaces, ODEs and Geometry

Text: Chapter 5	Lecture Notes: N/A	Slides: N/A

Quote of the Solutions to Homework One	
Homer: Do'h **Lisa**: A Deer. **Marge**: A female deer.	
	The Simpsons : Bart Gets an Elephant (S5E17, 1994)

1. Second Order Linear ODEs with Constant Coefficients

Often it is the case that ODEs appear in the form,

$$ay'' + by' + cy = f(x), \;\; a, b, c \in \mathbb{R}. \tag{1}$$

1.1. **Homogeneous Solution.** Solve the associated homogeneous problem.

Methods for solving this equation are known in detail. The reason for this is that the solution to the corresponding homogenous problem is totally understood and tractable by hand. The methods outlined here generalize to higher-order problems but lead to algebraic problems that require computational tools.[1]

Homogeneous Problems: Solving ODE's has relies heavily on the use of guessing. If you guess that $y_g(t)$ is a solution to an ODE then this assumption can be verified by direct substitution of the guess into the ODE. If equality is maintained then the guess is correct and if it isn't then the guess is incorrect. It turns out that the ODE,

$$ay'' + by' + cy = 0, \tag{2}$$

always leads to the same guess. The idea is that (2) is asking if there is exists a function such that differentiation of that function returns a constant multiple of the function itself. [2] The function with this property is the exponential function. [3] Substituting our guess, $y(t) = e^{\lambda t}$, into (2) gives,

$$ay'' + by' + cy = a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} \tag{3}$$

$$= (a\lambda^2 + b\lambda + c)e^{\lambda t} \tag{4}$$

$$= 0. \tag{5}$$

Since the exponential function is never zero we can divide it out of the equation and obtain,

$$a\lambda^2 + b\lambda + c = 0, \tag{6}$$

which is called the characteristic polynomial for the ODE. Since the polynomial is quadratic it can be solved in using the quadratic equation to get,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

When $b^2 - 4ac \neq 0$ the characteristic polynomial defines two linearly independent solutions to (2) and thus the complete homogeneous solution to the problem. When $b^2 - 4ac = 0$ then λ is a repeated-root and the only immediate solution is $y_1(t) = e^{-b/2a}$. However, a second solution can be found by the use of Theorem **??**.[4] These results are summarized in the following table.

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[1]We know that the characteristic polynomial must have n-many roots, counting multiplicity of course, past degree four it is known that they cannot, generally, be found by current analytic techniques. This result is known as Abel-Ruffini theorem and gives a point where numerical approximation must take over.

[2]Just think of the case where $a = 0, b = 1, c = -1$, which gives $y' = y$ whose solution is clearly $y(t) = e^t$. If we continue this line of thinking to $a = 1, b = -1, c = -2$ then the equation is $y'' = y' + 2y = \alpha y + 2y = (\alpha + 2)y$, which is again in the same form.

[3]Sine and cosine functions also share this property after more derivatives are taken. However, we won't worry about this since Euler's formula will find these functions for us.

[4]Direct substitution of $y_1(t) = e^{-b/2a}$ into Theorem **??** proves the common result that $y_2(t) = ty_1(t)$.

Discriminant	**Solutions**	**Homogeneous Solution**	**Definitions**
$b^2 - 4ac > 0$	$y_1(t) = e^{\lambda_1 t}$ $y_2(t) = e^{\lambda_1 t}$	$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$	$c_1, c_2 \in \mathbb{C}$ $\lambda_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ $\lambda_2 \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$
$b^2 - 4ac < 0$	$y_1(t) = e^{\lambda_1 t}$ $y_2(t) = e^{\lambda_1 t}$	$y_h(t) \quad = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ $= b_1 e^{\alpha t}\cos(\beta t) + b_2 e^{\alpha t}\sin(\beta t)$	$c_1, c_2, b_1, b_2 \in \mathbb{C}$ $b_1 = c_1 + ic_2,\ b_2 = c_1 - ic_2$ $\lambda = \alpha \pm \beta i$ $\alpha = \dfrac{-b}{2},\ \beta = \dfrac{\sqrt{4ac - b^2}}{2a}$
$b^2 - 4ac = 0$	$y_1(t) = e^{\lambda t}$ $y_2(t) = te^{\lambda t}$	$y_h(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}$	$c_1, c_2 \in \mathbb{C}$ $\lambda = \dfrac{-b}{2a}$

1.2. **Resonant Solutions.** A interesting prediction of the mathematics is resonant harmonic motion, which is an oscillatory solution whose amplitude grow in time. [5] Find the general solution the previous ODE when $a = c = 1$, $b = 0$ and $f(x) = \cos(x)$.

We know that $y_h(x) = c_1\cos(x) + c_2\sin(x)$ and must now guess $y_p(x) = Ax\cos(x) + Bx\sin(x)$. However, using, instead, a complex exponential we guess $y_p = Axe^{ix}$ take only the real part. Doing so gives,

$$y_p'' + y_p = 2iAe^{ix} = e^{ix}, \tag{8}$$

which implies that $A = 1/2i$ and $y_p(x) = Real\left\{Axe^{ix}\right\} = x\sin(x)/2$ and agrees with the result from class.

2. Power Series and Hyperbolic Trigonometric functions

Consider the ordinary differential equation:

$$y'' - y = 0 \tag{9}$$

2.1. **General Solution - Exponential Form.** Show that the solution to (9) is given by $y(x) = c_1 e^x + c_2 e^{-x}$.

2.2. **General Solution -Hyperbolic Form.** Show that $y(x) = b_1\sinh(x) + b_2\cosh(x)$ is a solution to (9) where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

2.3. **Conversion from Standard to Nonstandard Form.** Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1 e^x + c_2 e^{-x} = b_1\cosh(x) + b_2\sinh(x)$.

2.4. **Relation to Power-Series.** Assume that $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (9) in terms of the hyperbolic sine and cosine functions. [6]

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[5] These solutions are interesting in the sense that a bounded external force produces an unbounded solution. This has to do with the external force pumping energy into the system in 'just the right way.'

[6] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{10}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

please see second PDF for the solution to the previous problemplas

3. 2^{nd} Order Linear ODE: General Results

Typically, one arrives at the second-order linear ODE,

$$a(x)y'' + b(x)y' + c(x)y = f(x), \tag{11}$$

from Newton's or Kirchoff's law.

3.1. **Second Linearly Independent Solution.** Suppose that $a(x) = 1,\ b(x) = 4,\ c(x) = 4,\ f(x) = e^{-2x}$. [7] We know a solution to this problem is $y_1(x) = e^{-2x}$. Using the formula,

$$y_2(x) = k(x)y_1(x),\ k(x) = \int \frac{p(x)}{[y_1(x)]^2}dx,\ p(x) = e^{-\int (b(x)/a(x))dx}, \tag{12}$$

find a second linearly independent solution to the ODE.

We start by noting that, $p(x) = alpha_2 e^{-4x}$ where $\alpha_2 = e^{-\alpha_1},\ \alpha_1 \in \mathbb{R}$. Then $k(x) = \int dx = \alpha_2(x + \alpha_3) =$ and formally $y_2(x) = \alpha_2 x e^{-2x} + \alpha_2\alpha_3 e^{-2x}$. However, since $y_h(x) = \beta_1 y_1 + \beta_2 y_2$ we only need the linearly independent portion and $y_2(x) = xe^{-2x}$.

3.2. **Particular Solution: Part I.** Using the formula,

$$y_p(x) = y_2 \int \frac{f(x)y_1(x)}{W(x)}dx - y_1 \int \frac{f(x)y_2(x)}{W(x)}dx,\ W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x), \tag{13}$$

find the form for the particular solution. [8]

If notice that $W(x) = e^{-4x}$ then we have the particular solution as,

$$y_p(x) = y_2 \int dx - y_1 \int xdx \tag{14}$$

$$= \frac{x^2 e^{-2x}}{2} \tag{15}$$

3.3. **Particular Solution: Part II.** With our newfound trust, we use the previous formula on a problem that couldn't have been analyzed through previous methods. Solve the previous ODE where $a(x) = 1,\ b(x) = 0,\ c(x) = 1,\ f(x) = \sec(x)$, where $x > 0$.

Having solved $y'' + y = 0$ in class we quote the result $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$ and note $W(x) = 1$. Thus,

$$y_p(x) = y_2 \int dx - y_1 \int \tan(x)dx \tag{16}$$

$$= x\sin(x) - \cos(x)\ln|\cos(x)|. \tag{17}$$

4. Abstract Vector Spaces - Function Spaces

Given,

$$\left[m\frac{d^2}{dt^2} + k\right] y = 0,\ m, k \in \mathbb{R}^+ \tag{18}$$

$$\left[\frac{d}{dt} - \mathbf{A}\right] \mathbf{Y} = 0,\ \mathbf{A} \in \mathbb{R}^{2\times 2} \tag{19}$$

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[7] This problem is degenerate in the sense that it contains a repeated eigenvalue. Worse, the inhomogeneous term competes with the associated eigenfunction. You can solve this completely using techniques from your previous course work. We will use some formula to justify these techniques.

[8] You might notice that this can be done via the method of undetermined coefficients, which is considerably easier even if you have to multiply your 'guess' by two factors of x!

4.1. **Equivalence of Equations.** Find the change of variables that maps (18) onto (19) and using this define $\mathbf{Y}$ and $\mathbf{A}$. Consider the variable transformation defined by, $y' = v$. In this case, (18) becomes,

$$\frac{dv}{dt} = -\frac{k}{m}y, \tag{20}$$

and coupled with the transformation we have,

$$\frac{dy}{dt} = v, \tag{21}$$

$$\frac{dv}{dt} = -\frac{k}{m}y, \tag{22}$$

which is a linear system of differential equations where $\mathbf{Y}(t) = [y(t)\ v(t)]^{\mathrm{T}}$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}. \tag{23}$$

4.2. **Function Spaces.** Find the general solution to (19) and for $m = k = 1$ sketch its associated real phase-portrait.

In this case we have that $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = \lambda^2 + 1 = 0$. The eigenvalues are then $\lambda = \pm i$ and the eigenvectors are $\mathbf{Y} = [1\ \pm i]^{\mathrm{T}}$ and the general solution in real form is,

$$\mathbf{Y}(t) = c_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{24}$$

which is a clockwise rotation of the vector $\mathbf{c} = [c_1\ c_2]^{\mathrm{T}}$. Thus the phase-portrait is a clockwise parametrization of a circle of length $\sqrt{\mathbf{c}^{\mathrm{T}}\mathbf{c}}$.

5. Orthogonal Expansions

Given,

$$\hat{\mathbf{i}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{25}$$

5.1. **Orthonormality - Part I.** Show that the vectors are orthonormal by verifying the inner-products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = 1$.

We have,

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1 \tag{26}$$

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1 \tag{27}$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = -\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{2}{4} + \frac{2}{4} = 0 \tag{28}$$

5.2. **Orthogonal Representation I.** Show that any vector for $\mathbb{R}^2$ can be created as a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$. That is, given,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}, \tag{29}$$

show that c_1, c_2, can be found in terms of x_1 and x_2.

It is clear that this can be done via row-reduction but it is quicker using inner-products.

$$\hat{\mathbf{i}} \cdot \mathbf{x} = c_1\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + c_2\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} \implies c_1 = x_1\frac{\sqrt{2}}{2} + x_2\frac{\sqrt{2}}{2} \tag{30}$$

$$\hat{\mathbf{j}} \cdot \mathbf{x} = c_1\hat{\mathbf{j}} \cdot \hat{\mathbf{i}} + c_2\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} \implies c_2 = -x_1\frac{\sqrt{2}}{2} + x_2\frac{\sqrt{2}}{2} \tag{31}$$

5.3. **Orthonormality - Part II.** Show that $\langle f, g\rangle = (f,g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx$ satisfies the three axioms of a *Real Inner Product Space* and that $\langle \cos(nx), \cos(mx)\rangle = \langle \sin(nx), \sin(mx)\rangle = \pi\delta_{nm}$, $\langle \cos(nx), \sin(mx)\rangle = 0$ for all $n, m \in \mathbb{N}$.
First, we must agree that the space of integrable functions, V, satisfies the algebra/axioms of vector spaces, page 324, where $f(x) = 0$ and $g(x) = 1$ are the additive and multiplicative identities, respectively. With that said we now hope that,

$$\langle f, g\rangle = (f,g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx, \tag{32}$$

satisfies the axioms of a real inner-product space. Checking the three axioms for $c_1, c_2 \in \mathbb{R}$ and $f, g, h \in V$ we have,

(1) Axiom I : Linearity

$$\langle c_1 f + c_2 g, h\rangle = \int_{-\pi}^{\pi} \left[c_1 f(x) + c_2 g(x)\right] h(x)\,dx \tag{33}$$

$$= c_1 \int_{-\pi}^{\pi} f(x)h(x)\,dx + c_2 \int_{-\pi}^{\pi} g(x)h(x)\,dx \tag{34}$$

$$= c_1 \langle f, h\rangle + c_2 \langle g, h\rangle \tag{35}$$

(2) Axiom II : Symmetry

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx \tag{36}$$

$$= \int_{-\pi}^{\pi} g(x)f(x)\,dx \tag{37}$$

$$= \langle g, f\rangle \tag{38}$$

(3) Axiom III : Positive Definiteness
The following inner-product,

$$\langle f, f\rangle = \int_{-\pi}^{\pi} f(x)\cdot f(x)\,dx \tag{39}$$

$$= \int_{-\pi}^{\pi} \left[f(x)\right]^2\,dx, \tag{40}$$

is the integral of a non-negative function and implies that $\langle f, f\rangle \geq 0$. Since $f(x) = 0$ is the only function whose square contains zero area under its curve we also conclude that $\langle f, f\rangle = 0 \iff f(x) = 0$.

The previous results show that the space of integrable functions is also a real inner-product space with inner-product defined as the previous integral. These integrals will be very important throughout the study of Fourier series and PDE and is an archetype of an orthogonality argument for abstract real inner-product spaces. The result listed above is what we will use but first we must justify the equality. To do we we consider the following argument for $n, m \in \mathbb{Z}$,

$$\left\langle e^{inx}, e^{\pm imx}\right\rangle = \int_{-\pi}^{\pi} e^{inx}e^{\pm imx}\,dx \tag{41}$$

$$= \int_{-\pi}^{\pi} e^{i(n\pm m)x}\,dx \tag{42}$$

$$= i(n\mp m)^{-1}\, e^{i(n\pm m)x}\Big|_{-\pi}^{\pi} \tag{43}$$

$$= i(n\mp m)^{-1}\left[e^{i(n\pm m)\pi} - e^{-i(n\pm m)\pi}\right] \tag{44}$$

$$= i(n\mp m)^{-1}\left[(-1)^{n\pm m} - (-1)^{n\pm m}\right] \tag{45}$$

$$= \begin{cases} 0, & \text{for } e^{imx} \text{ and any } n, m \\ 0, & \text{for } e^{-imx} \text{ and } n \neq m \end{cases}. \tag{46}$$

To relate this to the integral in question we note that,

$$\text{Real}\left[\left\langle e^{inx}, e^{\pm imx}\right\rangle\right] = \int_{-\pi}^{\pi} \left[\cos(nx)\cos(mx) \mp \sin(nx)\sin(mx)\right]\,dx = 0 \implies \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \pm\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx, \tag{47}$$

which implies that each integral is zero. [9] When $n = m$ it is quick to verify $\left\langle e^{inx}, e^{-inx} \right\rangle = 2\pi$. Using the previous relations we have,

$$\text{Real}\left[\left\langle e^{inx}, e^{-inx} \right\rangle\right] = \int_{-\pi}^{\pi} \left[\cos(nx)\cos(nx) + \sin(nx)\sin(nx)\right]\, dx \tag{48}$$

$$= 2\int_{-\pi}^{\pi} \sin(nx)\sin(nx)\, dx \tag{49}$$

$$= 2\pi. \tag{50}$$

Taken together we have the desired results, $\langle \cos(nx), \cos(mx) \rangle = \langle \sin(nx), \sin(mx) \rangle = \pi\delta_{nm}$. The remaining result $\langle \cos(nx), \sin(mx) \rangle = 0$ can be easily shown through symmetry.

5.4. **Orthogonal Representation II.** Show that if $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$ then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \tag{51}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \tag{52}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{53}$$

The idea is the same as above. We note the following integral relations,

$$\langle \sin(nx), \sin(mx) \rangle = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \pi\delta_{nm}, \tag{54}$$

$$\langle \cos(nx), \cos(mx) \rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \pi\delta_{nm}, \tag{55}$$

$$\langle \sin(nx), \cos(mx) \rangle = \int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0, \tag{56}$$

where $n, m \in \mathbb{N}$. Now using the same idea as the previous problem we have for fixed m,

$$\langle \sin(mx), f(x) \rangle = \left\langle \sin(mx), a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right\rangle \tag{57}$$

$$= \langle \sin(mx), a_0 \rangle + \sum_{n=1}^{\infty} a_n \langle \sin(mx), \cos(nx) \rangle + b_n \langle \sin(mx), \sin(nx) \rangle \tag{58}$$

$$= \sum_{n=1}^{\infty} b_n \pi \delta_{nm} \tag{59}$$

$$= b_m \pi \delta_{mm} \implies b_m = \frac{1}{\pi}\langle \sin(mx), f(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx, \tag{60}$$

and

$$\langle \cos(mx), f(x) \rangle = \left\langle \cos(mx), a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right\rangle \tag{61}$$

$$= \langle \cos(mx), a_0 \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(mx), \cos(nx) \rangle + b_n \langle \cos(mx), \sin(nx) \rangle \tag{62}$$

$$= \sum_{n=1}^{\infty} a_n \pi \delta_{nm} \tag{63}$$

$$= a_m \pi \delta_{mm} \implies a_m = \frac{1}{\pi}\langle \cos(mx), f(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx, \tag{64}$$

[9] The only number such that $A = \pm A$ is true is the number zero.

and lastly,

$$\langle 1, f(x)\rangle = \left\langle 1, a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \right\rangle \tag{65}$$

$$= \langle 1, a_0\rangle + \sum_{n=1}^{\infty} a_n \langle 1, \cos(nx)\rangle + b_n \langle 1, \sin(nx)\rangle \tag{66}$$

$$= \langle 1, a_0\rangle \tag{67}$$

$$= 2\pi a_0 \implies a_0 = \frac{1}{\pi} \langle 1, f(x)\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx. \tag{68}$$

Since m is arbitrary we can replace m with n and recover the desired result.