INTRODUCTION TO GENERALIZED FUNCTIONS

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ABSTRACT. The following notes attempt to shed some light on the so-called Dirac delta function. This 'function' is generally presented to students studying Laplace transforms. The level of formality differs but at the end of the day the Dirac delta 'function' is adopted as a tool for use within problem at hand. This is very much our case and though the mathematical rigor will not really add much to the tool, it is nevertheless important to pay some homage to it and in doing so highlight the field of mathematics in which it is placed. Thus, it is important to say that the goal of these notes is contextualization and not rigorous justification of the mathematics. Those students who want to peak behind the curtain should study more mathematics but it is important for everyone exposed to such mathemagic to feel less duped.

1. INTRODUCTION

Those that have been shown the so-called Dirac delta 'function' should recognize the following properties:

P1. Ideal Localization: The Dirac function is zero except for a single point on the $x-$axis. That is, $\delta(x-x_0)=0$ for all $x\neq x_0\in\mathbb{R}$.

P2. Unit Impulse: Whatever the Dirac function represents, the area under its 'curve' is one. That is, $\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)dx=1$ for all $\epsilon>0$.[1]

P3. Ideal Measurement: Integration with the Delta function selects one value of the function that it is integrated against. That is, $\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)f(x)dx=f(x_0)$ for integrable f and all $\epsilon>0$.[2]

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[1] This is where the train goes off the tracks. At this point you are somehow made to believe that there is a unit of area contained in what is essentially an line segment. If this is not convincing then you must somehow justify that the Delta function gets large enough to make this okay. There is no way that this is okay. Really. However, to make sense out of this statement I offer the following. Suppose that the Delta function is an external force to a mass-spring system, $f(t)=\delta(t)$. This says that you are forcing the system at the start of time for an instantaneous moment and this is really where the problem comes from. Can we really do anything in an instant? Well, the answer is, of course, no we cannot. Now, if we recall the relationship $my''+ky=0 \iff \frac{m}{2}v^2+\frac{k}{2}y^2=E$ is given through integrating the ODE then we might think to do the same trick for the forced case. Doing so, we would have to integrate the Delta function against the velocity, which would give one-unit of area under its curve multiplied by the initial velocity. What is this a unit of? Well, it would be such that it times velocity is energy. So, the Delta function is the idealized notion that we can give a finite amount of 'energy' to a system in an infinitesimal amount of time. Totally ideal but we see why we might want such an idealization.

[2] The idea of measurement is important will be elaborated on later in these notes.

Seeking to rigorously justify the above statements, we introduce the concept of a delta sequence through the use of the so-called Heavyside function,

$$H(x-x_0)=\left\{\begin{array}{ll} 0, & x<x_0, \\ 1, & x\geq x_0 \end{array}\right. , \tag{1}$$

where $x_0 \in \mathbb{R}$. This function is also called the unit-step function and is often used to turn forces on and off in problems from differential equations. Using this function we define

$$\delta_a(x-x_0)=\frac{1}{2a}\left[H(x-(x_0-a))-H(x-(x_0+a))\right], \tag{2}$$

for some $a \in \mathbb{R}$, which is called a delta sequence. Plotting this function gives a non-zero horizontal line that is $2a$ units wide and $(2a)^{-1}$ units tall. Thus, this curve has one-unit of area under its curve for all a.

If we think of the non-zero portion of the graph as the information/disturbance then the idea of Dirac is to localize this non-trivial part as much as possible. This *idealization* is called the Dirac delta distribution/function,

$$\delta(x-x_0)=\lim_{a\to 0}\delta_a(x-x_0). \tag{3}$$

Clearly, the Dirac delta function has the property that $\delta(x-x_0)=0$ for all $x\neq x_0$.[3] Thus, to show the other properties we have,

$$\lim_{a\to 0}\int_{-\infty}^{\infty}\delta_a(x-x_0)f(x)dx = \lim_{a\to 0}\frac{1}{2a}\left[\int_{-\infty}^{\infty}H(x-(x_0-a))f(x)dx-\int_{-\infty}^{\infty}H(x-(x_0+a)f(x)dx\right] \tag{4}$$

$$= \lim_{a\to 0}\frac{1}{2a}\left[\int_{x_0-a}^{\infty}f(x)dx-\int_{x_0+a}^{\infty}f(x)dx\right] \tag{5}$$

$$= \lim_{a\to 0}\frac{1}{2a}\left[\lim_{x\to\infty}F(x)-F(x_0-a)-\lim_{x\to\infty}F(x)+F(x_0+a)\right] \tag{6}$$

$$= \lim_{a\to 0}\frac{1}{2a}\left[F(x_0+a)-F(x_0-a)\right] \tag{7}$$

$$\stackrel{h}{=} \lim_{a\to 0}\frac{f(x_0+a)+f(x_0-a)}{2} \tag{8}$$

$$= f(x_0), \tag{9}$$

which is the desired result, where F is the anti-derivative of f and $\stackrel{h}{=}$ indicates the use of L'Hôpital's rule. This gives property three listed above and if $f(x)=1$ then we have recovered property two. That said, by being rigorous here, we have only invited more questions. Namely,

Q1. If $\delta(x-x_0)=\lim_{a\to 0}\delta(x-x_0)$ then can we consider
$$\lim_{a\to 0}\int_{-\infty}^{\infty}\delta_a(x-x_0)dx=\int_{-\infty}^{\infty}\delta(x-x_0)dx?$$
Q2. What is $\delta(x-x_0)$?

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[3] In fact, when $x=x_0$ we should have $\delta(0)=\infty$ however, this is violates our definition of function, which says that it should take in numbers from $\mathbb{R}$ and return numbers in $\mathbb{R}$ and $\infty\notin\mathbb{R}$. Remember, what we are dealing with is an idealization and there is no reason to think that upon taking this limit we have a function at all.

The second of these questions is really at the heart of the matter and the first only highlights the difficulty of giving meaning to the Dirac delta function. Specifically, if we take the Dirac delta function to be nontrivial only at a single point in space then at best the Dirac delta function encapsulates a line segment, which we must admit has no area whatsoever and the integrals from the first question contradict one another. That is, we write, $\int_{-\infty}^{\infty} \delta(x-x_0)dx$, which we know must be zero, but really we mean $\lim_{a\to 0}\int_{-\infty}^{\infty} \delta_a(x-x_0)dx$, which we have shown is non-zero. More importantly, $\delta(x-x_0) = \lim_{a\to 0}\delta(x-x_0)$ has no intrinsic meaning because $\delta(x)$ is not a function. Moreover, any reasonable definition of integration prohibits the existence of a function that has all three of the original properties.

So, what is the meaning of $\delta(x)$?

> **Key Point**:
> Mathematically, we must understand the Dirac delta function to be the limit of a sequence of integrals such that property 3 is obeyed for every reasonable choice of f. [4]

However, this does not give a lot of meaning to the Dirac delta function, which is our goal. Meaning comes from the theory of distributions, which is discussed in the next section.

2. Concepts from Distribution Theory

3. Conclusions

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[4] Clearly, P3 recovers P2 and at this point we agree to never speak of the mathematics of P1 again.