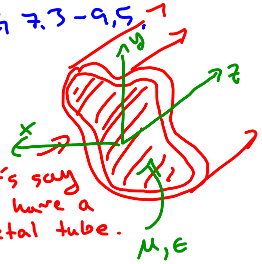
Readin Today: G 9.5

Next: Review G 7.3-9.5

Assume perfectly conducting walls,

$\Rightarrow$ Just inside the walls

$\vec{E}_{\parallel} = \emptyset$; $B_{\perp} = \emptyset$



Let's say we have a metal tube.

Maxwell's eqns inside waveguide.

① $\vec{\nabla}\cdot\vec{E} = \emptyset$ ③ $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

② $\vec{\nabla}\cdot\vec{B} = \emptyset$ ④ $\vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$

If propagation is in z-direction then

$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$ $\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$ E_0, B_0 depend only on x, y

③ $\hat{z}$: $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = +i\omega B_z$ ④ $\hat{z}$: $\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{v^2}E_z$

$\hat{x}$: $\frac{\partial E_z}{\partial y} - \underset{ikE_y}{\frac{\partial E_y}{\partial z}} = +i\omega B_x$ $\hat{x}$: $\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{v^2}E_x$

$\hat{y}$: $\underset{ikE_x}{\frac{\partial E_x}{\partial z}} - \frac{\partial E_z}{\partial x} = +i\omega B_y$ $\hat{y}$: $ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{v^2}E_y$

$$ikE_x - \frac{\partial E_z}{\partial x} = \frac{i\omega}{ik}\left[\frac{\partial B_z}{\partial y} + \frac{i\omega}{v^2}E_x\right]$$

$$\Rightarrow -k^2E_x - ik\frac{\partial E_z}{\partial x} = i\omega\frac{\partial B_z}{\partial y} - \frac{\omega^2}{v^2}E_x$$

$$E_x = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

Also you can get

$$E_y = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{v^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{v^2}\frac{\partial E_z}{\partial x}\right)$$

$$E_x = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

Also you can get

$$E_y = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{v^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{\omega^2/v^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{v^2}\frac{\partial E_z}{\partial x}\right)$$

lets define $\vec{E}_T = E_x\hat{x} + E_y\hat{y} \Rightarrow \vec{E} = \vec{E}_T + E_z\hat{z}$

$\vec{B}_T = B_x\hat{x} + B_y\hat{y} \Rightarrow \vec{B} = \vec{B}_T + B_z\hat{z}$

$$\vec{E}_T = \frac{i}{\omega^2/v^2 - k^2}\left[k\left[\frac{\partial E_z}{\partial x}\hat{x} + \frac{\partial E_z}{\partial y}\hat{y}\right] + \omega\left[\frac{\partial B_z}{\partial y}\hat{x} - \frac{\partial B_z}{\partial x}\hat{y}\right]\right]$$

lets define an operator $\vec{\nabla}_T = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$

$\vec{\nabla} = \vec{\nabla}_T + \vec{\nabla}_z$

Show $\vec{\nabla}_T \cdot \vec{\nabla}_T f = \nabla_T^2 f$ $\left\{\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}$

$\vec{\nabla}_T \cdot (\vec{\nabla}_T \times \vec{v}) = \emptyset$

Rewriting $\vec{E}_T$ and $\vec{B}_T$

$$\vec{E}_T = \frac{i}{\frac{\omega^2}{v^2} - k^2}\left[k\vec{\nabla}_T E_z + \omega\vec{\nabla}_T \times \vec{B}_z\right]$$

$$\vec{B}_T = \frac{i}{\omega^2/v^2 - k^2}\left[k\vec{\nabla}_T B_z - \frac{\omega}{v^2}\vec{\nabla}_T \times \vec{E}_z\right]$$

Max ①: $\vec{\nabla}\cdot\vec{E} = \emptyset$ $\quad \vec{\nabla}_T \cdot \vec{E}_T + ikE_z = \emptyset$

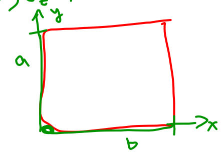
Find a diff eq. for E_z using and .

$$\vec{\nabla}_T \cdot \left[\frac{i}{\omega^2/v^2 - k^2}\left(k\vec{\nabla}_T E_z + \omega\cancel{\vec{\nabla}_T \times B_z}^{\,\emptyset}\right)\right] + ikE_z = \emptyset$$

$$\nabla_T^2 E_z + (\omega^2/v^2 - k^2)E_z = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (\omega^2/v^2 - k^2)E_z = \emptyset$$

Same for B_z $\quad \frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + (\omega^2/v^2 - k^2)B_z = \emptyset$



Aside: TE (transverse electric)

$\Rightarrow E_z = \emptyset$.

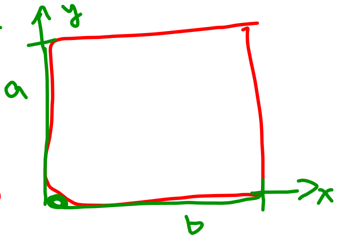
TM (" magnetic)

$\Rightarrow B_z = \emptyset$

TEM $\Rightarrow E_z = B_z = \emptyset$.

The book treats the TE case.

Lets start the TM case.



Separation of variables!

Assume $E_z = X(x)Y(y)$

$$\frac{\partial^2}{\partial x^2}XY + \frac{\partial^2}{\partial y^2}XY + \left(\frac{\omega^2}{v^2}-k^2\right)XY = 0$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \left(\frac{\omega^2}{v^2}-k^2\right)XY = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \left(\frac{\omega^2}{v^2}-k^2\right) = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \;;\; \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$$

$$X = A_1\cos(k_x x) + A_2\sin(k_x x) \qquad Y = B_1\cos(k_y y) + B_2\sin(k_y y)$$

$\vec{E}_\parallel = 0 \Rightarrow E_z = 0$ on the boundarys.

$$\Rightarrow A_1 = B_1 = 0$$

$$X = A_2\sin(k_x x) \qquad Y = B_2\sin(k_y y)$$

$$\frac{\omega^2}{v^2} - k^2 - k_x^2 - k_y^2 = 0$$

$$k_x^2 + k_y^2 + k^2 = \frac{\omega^2}{v^2}$$

$k_x = \frac{n\pi}{b}$ $\quad k_y = \frac{m\pi}{a}$ $\quad$ {n, m are integers}

Because there's an upper limit to $k_x^2 + k_y^2 < \frac{\omega^2}{v^2}$ then there is a finite # of propagating modes.