2 _ 13 _ 08

Note Title 2/12/2008

Tricky variance issue.

Population: e.g., all CSM students
means + variances unknown a priori

Sample population: N students chosen at random.

We want to use the sample population to estimate the variance of some property (e.g. height) of the population

Actual, unknown, variance

$$\sigma^2 = \sum_{i=1}^{N_{pop}} (h_i - \bar{h}_i) P_r[h_i]$$

2 different estimators:

$$S_N^2 = \frac{1}{N} \sum_{i=1}^{N} (h_i - \bar{h}_i)^2$$

$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (h_i - \bar{h}_i)^2$$

obviously $s_N^2 \approx s^2$ for large N.

But only s^2 is an unbiased estimator of σ^2. I.e.,

$$E[s^2] = \sigma^2$$

expected value

proof in any prob/stat book.

data = $\{3, 1, 4, 1, 5, 9, \ldots\}$ 25 digits

Mathematica's built in standard deviation computes $s = 2.52521$ whereas $s_N = 2.47419$

$$\frac{s}{s_N} = .979796 = \sqrt{\frac{24}{25}}$$

if you lost 1/2 point on 1.10.c talk to Jared

See note from 2/11/08 we got solution away from origin:

$$\psi(x) = B e^{-|x|}$$

Near the origin TISE

$$-\frac{\hbar^2}{2m}\psi''(x) - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x)$$

Remember: the δ is just a mathemat. idealization of a deep, narrow well.

But since we're using it we have to be careful.

$$\alpha\,\delta(x)\,\psi(x)$$

only makes mathematical sense if we integrate it.

$$-\frac{\hbar^2}{2m}\underbrace{\int_{-\epsilon}^{\epsilon}\psi''\,dx}_{} - \alpha\underbrace{\int_{-\epsilon}^{\epsilon}\delta(x)\,\psi(x)\,dx}_{} = E\int_{-\epsilon}^{\epsilon}\psi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} - \alpha\,\psi(0) = E\,\psi(a)\,2\epsilon$$

1) ψ is continuous
2) $d\psi/dx$ is continuous, except at points where $V(x)$ is infinite

So becomes

$$\left.\left(\frac{d\psi}{dx}\right)\right|_{-\epsilon}^{\epsilon} = -\frac{2m\alpha}{\hbar^2}\,\psi(0)$$

Now, except at $x=0$ $\quad \psi(x) = Be^{-\kappa|x|}$

so for $\quad x < 0 \quad \psi(x) = Be^{\kappa x}$

$\qquad\quad x > 0 \quad \psi(x) = Be^{-\kappa x}$

$\Rightarrow \quad x < 0 \quad \psi' = \kappa B e^{\kappa x}$

$\qquad\quad x > 0 \quad \psi' = -\kappa B e^{-\kappa x}$

So provided ϵ is small

$$\psi'_{+\epsilon} - \psi'_{-\epsilon} = -2B\kappa$$

$$\psi(0) = B$$

$$\Rightarrow -2B\kappa = \frac{-2m\alpha}{\hbar^2} B$$

$$\Rightarrow \boxed{\kappa = \frac{m\alpha}{\hbar^2}}$$

And since $\quad \kappa \equiv \dfrac{\sqrt{-2mE}}{\hbar}$

$$\Rightarrow \frac{m^2\alpha^2}{\hbar^4} = \frac{-2mE}{\hbar^2} \Rightarrow \boxed{E = -\frac{1}{2}\frac{m\alpha^2}{\hbar^2}}$$

Finally we need to normalize ψ

$$\psi(x) = B e^{-\kappa|x|} = B e^{-m\alpha/\hbar^2 |x|}$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = B^2 \int_{-\infty}^{\infty} e^{-\frac{2m\alpha}{\hbar^2}|x|} dx$$

$$= B^2 \frac{\hbar^2}{m\alpha} = 1$$

$$\boxed{B = \frac{\sqrt{m\alpha}}{\hbar}}$$

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} \qquad E = -\frac{m\alpha^2}{2\hbar^2}$$

<u>There is exactly one bound state</u>

For Scattering states $E > 0$

Same procedure but

$$\psi'' = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$$

$$k^2 = \frac{2mE}{\hbar^2}$$

So away from the origin:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \qquad x < 0$$

$$\psi(x) = Fe^{ikx} + Ge^{-ikx} \qquad x > 0$$

$$\psi(x_-) = \psi(x_+) \qquad \text{(continuity of } \psi\text{)}$$

$$\Rightarrow \quad A + B = F + G$$

$$\psi'(x) = ik\left(Ae^{ikx} - Be^{-ikx}\right) \qquad x < 0$$
$$\psi'(x) = ik\left(Fe^{ikx} - Ge^{-ikx}\right) \qquad x > 0$$

$$\psi'\Big|_{x=-\epsilon}^{x=+\epsilon} = ik\left[F-G\right] - ik\left[A-B\right]$$

$$= ik\left[F - G - A + B\right]$$

$$\psi(0) = A+B = F+G$$

As before

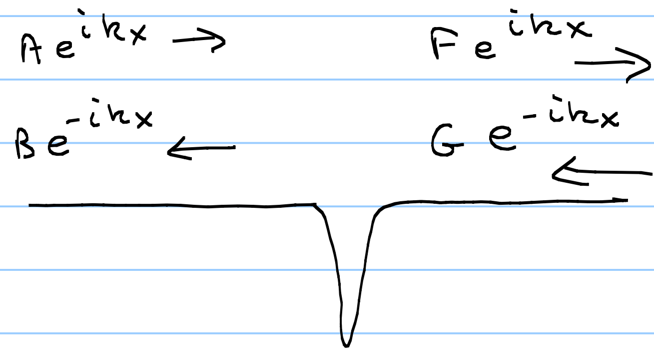
$$\left(\frac{d\psi}{dx}\right)\Big|_{-\epsilon}^{\epsilon} = -\frac{2m\alpha}{\hbar^2}\,\psi(0)$$

$$ik\left[F-G-A+B\right] = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$\Rightarrow F - G = \frac{2im\alpha}{k\hbar^2}(A+B) + A - B$$

$$F - G = 2i\beta(A+B) + A - B$$

Can't use normalization here (why?)



Suppose we consider the case of a plane wave incident from left

This means that $G = 0$

So $$F = A + B$$
$$F = 2i\beta(A+B) + A - B$$

$$A + B = 2i\beta(A+B) + A - B$$

$$2B = 2i\beta(A+B)$$

$$2(1-i\beta)B = 2i\beta A$$

$$\boxed{B = \frac{i\beta}{1-i\beta} A}$$

Then, since $F = A + B$

$$F = \left(1 + \frac{i\beta}{1-i\beta}\right) A$$

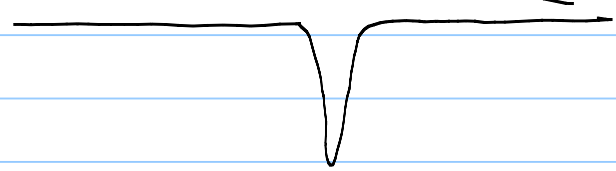
$$\Rightarrow \quad F = \frac{1 - i\beta \quad + i\beta}{1 - i\beta} A$$

$$\boxed{F = \frac{1}{1 - i\beta} A}$$

Recall

$Ae^{ikx} \rightarrow$ $\qquad$ $Fe^{ikx} \rightarrow$

$Be^{-ikx} \leftarrow$ $\qquad$ $Ge^{-ikx} \leftarrow$ $\qquad$ $G = 0$



So $\dfrac{|B|^2}{|A|^2}$ is the reflection coeff.

And $\dfrac{|F|^2}{|A|^2}$ is the transmission Coef.

$$T = \frac{|F|^2}{|A|^2} = \left|\frac{1}{1 - i\beta}\right|^2 = \frac{1}{1+\beta^2}$$

$$R = \frac{|B|^2}{|A|^2} = \left|\frac{i\beta}{1 - i\beta}\right|^2 = \frac{\beta^2}{1+\beta^2}$$

Notice that $|T|^2 + |R|^2 = 1$.

$$\beta = \frac{m\alpha}{k\hbar^2} \qquad k^2 = \frac{2mE}{\hbar^2}$$

$$\beta^2 = \frac{m^2\alpha^2}{k^2\hbar^4} = \frac{m^2\alpha^2}{\frac{2mE}{\hbar^2}\cdot\hbar^4} = \frac{\frac{1}{2}m\alpha^2}{E\hbar^2}$$

$$T = \frac{1}{1 + m\alpha^2/(2E\hbar^2)}$$

$$R = \frac{1}{1 + (2\hbar^2 E/m\alpha^2)}$$

NB if we make $\alpha \to -\alpha$ the well becomes a barrier but T + R are unchanged since they depend on α^2.

But there are no bound states now.