Lecture 2: Boundary conditions in statics

Let's back up a bit and consider static problems. Then:

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = 0 \qquad \nabla\times\vec{B} = \mu_0\vec{J}$$

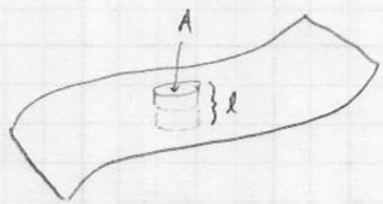
Many E+M problems are of the form "Given these sources, find the fields."

The Maxwell equations are the partial differential equations that let you do just that. You may recall that solving PDEs requires boundary conditions, and in a not-quite-circular way, the Maxwell equations each produce a generally useful BC.

Consider Gauss's law: $\nabla\cdot\vec{E} = \rho/\varepsilon_0$

Or, in integral form, $\oint \vec{E}\cdot d\vec{A} = Q_{enc}/\varepsilon_0$

Now consider an arbitrary boundary, some kind of surface:



Draw a little Gaussian box so as to enclose part of that surface, with endcaps parallel to our boundary. Now shrink the sides down ($l\to 0$) in the picture. Then the endcaps of area A are the only chunks with finite area, and given some $\vec{E}$ in the neighborhood, Gauss's law becomes

$$\oint \vec{E}\cdot d\vec{A} = \int_{top} \vec{E}_{above}\cdot d\vec{A} + \int_{bottom} \vec{E}_{below}\cdot d\vec{A} = Q_{enc}/\varepsilon_0$$

With the convention that outward is the positive direction for area vectors (so $d\vec{A}$ for top & bottom are opposite signs), we get

$$\int_{top} E_{\perp,above}\,dA - \int_{bot} E_{\perp,bot}\,dA = Q_{enc}/\varepsilon_0 = \int \frac{\sigma}{\varepsilon_0} dA$$

Where $E_\perp$ is the component of E perpendicular to our surface.

Over small areas, E is effectively constant, so we get

$$E_{\perp,above}A - E_{\perp,below}A = \frac{\sigma}{\varepsilon_0}A \Rightarrow \boxed{E_{\perp,above} - E_{\perp,below} = \frac{\sigma}{\varepsilon_0}}$$

That's a big deal. In words, it says:

At any boundary surface, the perpendicular component of the E-field is either continuous if there is no charge, $(E_{\perp,a} - E_{\perp,b} = 0)$

or discontinuous by an amount related to the local charge density.

This is consistent with the "field made by an infinite sheet" situation

$$\begin{array}{c} \uparrow\uparrow\uparrow \ |\vec{E}| = \sigma/2\varepsilon_0 \\ \hline \downarrow\downarrow\downarrow \ E = \sigma/2\varepsilon_0 \end{array} \ \sigma$$

but much more general.

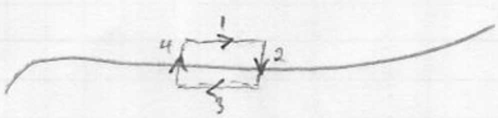
Identical steps take you from $\nabla\cdot\vec{B} = 0$ to $\boxed{B_{\perp,above} - B_{\perp,below} = 0}$

Which is to say, the component of B perpendicular to some boundary is always continuous. Always.

In fact, anytime you know the divergence of a vector field, you can do this stuff to pop out a BC on the perpendicular field component.

Does something useful happen if you know the curl of a vector field? Yes, it does.

Consider a boundary once again.



Draw a little loop that encloses the boundary and look at $\oint \vec{E}\cdot d\vec{l}$ along that loop.

Since $\nabla\times\vec{E} = 0$, $\oint \vec{E}\cdot d\vec{l} = 0$

Let 2 + 4 shrink down to infinitesimals and let 1 + 3 get short enough that E is locally constant. Then

$$\oint \vec{E}\cdot d\vec{l} = \vec{E}_1\cdot\vec{l}_1 + \vec{E}_3\cdot\vec{l}_3 = 0$$

Since $\vec{l}_1$ + $\vec{l}_3$ are in opposite directions,

$\oint \vec{E}\cdot d\vec{l} = E_{1,\parallel} l_1 - E_{3,\parallel} l_3 = 0$

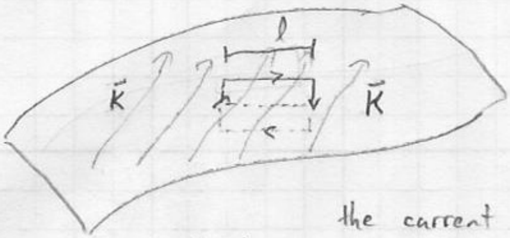
$\Rightarrow \quad E_{\parallel,above} - E_{\parallel,below} = 0$

$\Rightarrow \quad \boxed{E_{\parallel,above} = E_{\parallel,below}}$

With $E_\parallel$ indicating the component of the E-field parallel to the boundary.

So the component of $\vec{E}$ parallel to some boundary is <u>always</u> continuous in statics.

There's one left: $\nabla\times\vec{B} = \mu_0 \vec{J}$



Here's a boundary with some sheet current $\vec{K}$ flowing on it.

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$, where I_{enc} is the current flowing through the loop.

So here, $I_{enc} = K_\perp l$, where $K_\perp$ is the component of $\vec{K}$ normal to the surface defined by the loop. Skipping forward,

$B_{1,\parallel} l - B_{3,\parallel} l = \mu_0 K_\perp l \quad \Rightarrow \quad B_{1,\parallel} - B_{3,\parallel} = \mu_0 K_\perp$

Which we can express in vector form as

$\boxed{\vec{B}_{\parallel,above} - \vec{B}_{\parallel,below} = \mu_0 \vec{K}\times\hat{n}}$

With $\hat{n}$ the unit vector normal to the surface (not the loop)

$\vec{K}\times\hat{n}$ points in a direction $\perp$ to both $\vec{K}$ + the surface normal, and so is the direction along which you'd put the loop to enclose all of $\vec{K}$.

So $\vec{B}_\parallel$ is always continuous except across surface currents.

One last, rather useful result: Since $\vec{E} = -\nabla V$ and $\vec{E}$ has to be finite always, V always has finite derivatives. This means V is <u>always</u> continuous.

Putting it all together, in statics we have a set of <u>totally general</u> boundary conditions:

$E_{1,\perp} - E_{2,\perp} = \sigma/\varepsilon_0 \qquad \vec{B}_{1,\parallel} - \vec{B}_{2,\parallel} = \mu_0 \vec{K}\times\hat{n}$

$B_\perp$, $E_\parallel$, and V always continuous.