MATH332-Linear Algebra	Homework Three

Linear Transformations and Matrix Algebra

Text: 1.7-2.2 Section Overviews: 1.7-2.2

Quote of Homework Three	
Doc Platter: I never could figure what the sky was thinking but the soil she don't keep too many secrets.	
	King of the Hill : A Rover Runs through it S9E1 (2004)

1. Review and Warm Ups

1.1. **Parameterized Vector Form.** Given the following non-homogeneous linear system,

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6. \end{aligned}$$

Describe the solution set of the previous system in parametric vector form, and provide a geometric comparison with the solution to the corresponding homogeneous system.

1.2. **Spanning Sets.** Given,

$$\mathbf{A} = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}. \tag{1}$$

Do the columns of $\mathbf{A}$ form a linearly independent set? What is the spanning set of the columns of $\mathbf{A}$?

1.3. **Range of a Matrix Transformation.** Given,

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2}$$

Show that there does not exist a solution to $\mathbf{Ax} = \mathbf{b}$ for every $\mathbf{b} \in \mathbb{R}^3$ and describe the set of all $\{b_1, b_2, b_3\}$ for which $\mathbf{Ax} = \mathbf{b}$ *does* have a solution.

2. Continued Work with Language

Answer each true/false question in the chapter 1 supplemental section on page 102. It is not necessary to supply justifications but if you want your logic checked then feel free to provide a justification.

3. Theory

3.1. **Characterization of a Null Mapping.** Suppose the vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose, as well, that $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \ldots, p$. Show that T is the zero-transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

3.2. **Underdetermined and Square Matrix Transformations.** If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ then can you give a relation between m and n? What if T is also a one-to-one transformation?

3.3. **Transforming Linearly Combinations.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vectors onto a linearly dependent set of vectors, then the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.[1]

[1]Hint: Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are linearly independent but $T(\mathbf{u}), T(\mathbf{v}) \in \mathbb{R}^m$ are not. First, what must be true of $\mathbf{u} + \mathbf{v}$? Also, what must be true of c_1, c_2 for $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$? Using these facts show that $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

Given,

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

4.1. **Surjective Mapping.** Show that this transformation is onto $\mathbb{R}^2$. [2]

4.2. **Injective Mapping.** Show that the transformation is one-to-one. [3]

4.3. **The Unit Circle.** Show that the transformation $\mathbf{A}\hat{\mathbf{i}}$ rotates $\hat{\mathbf{i}} = [1\ 0]^{\mathrm{T}}$ counter-clockwise by an angle θ and defines a parametrization of the *unit circle*. What matrix would undo this transformation?

4.4. **Determinant.** Show that $\det(\mathbf{A}) = 1$.[4]

4.5. **Orthogonality.** Show that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$.

4.6. **Classical Result.** Let $\mathbf{A}(\theta)\mathbf{x} = \mathbf{b}$ for each $\theta \in S$. Calculate, $\dfrac{\mathbf{x}\cdot\mathbf{b}}{|\mathbf{x}||\mathbf{b}|}$.[5] How is this related to θ? [6]

4.7. **Differentiation of Matrix Functions.** If we define the derivative of a matrix function as a matrix of derivatives then a typical product rule results. That is, if $\mathbf{A}, \mathbf{B}$ have elements, which are functions of the variable θ then $\dfrac{d}{d\theta}[\mathbf{AB}] = \mathbf{A}\dfrac{d\mathbf{B}}{d\theta} + \dfrac{d\mathbf{A}}{d\theta}\mathbf{B}$. [7] Using the identity $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, show that

4.8. **Rotations in $\mathbb{R}^3$.** Given, $\dfrac{d\left[\mathbf{A}^{-1}\right]}{d\theta} = \mathbf{A}^{-1}\dfrac{d\mathbf{A}}{d\theta}\mathbf{A}^{-1}$. Verify this formula using the $\mathbf{A}$ matrix given above.

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad \mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Describe the transformations defined by each of these matrices on vectors in $\mathbb{R}^3$.

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[2]Recall that a transformation is onto if there exists an $\mathbf{x}$ for every $\mathbf{b}$ in the co-domain.

[3]Recall that a transformation is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

[4]We haven't discussed determinants yet but for 2×2 there is an easy formula given by,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies \det(\mathbf{A}) = ad - bc \tag{3}$$

, which should look familiar from homework 2.1.2 associated with requirement for having only the trivial solution.

[5]Recall that $\mathbf{x}\cdot\mathbf{y}$ and $|\mathbf{x}|$ are the standard dot-product and magnitude, respectively, from vector-calculus. These operations hold for vectors in $\mathbb{R}^n$ but now have the following definitions, $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ and $|\mathbf{x} = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$.

[6]What we are trying to extract here is the standard result from calculus, which relates the dot-product or inner-product on vectors to the angle between them. This is clear when we have vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ since we have tools from trigonometry and geometry but when treating vectors in $\mathbb{R}^n, n \geq 4$ these tools are no longer available. However, we would still like to have similar results to those of $\mathbb{R}^n, n = 2, 3$. To make a long story short, we will have these results for arbitrary vectors in $\mathbb{R}^n$ but not immediately. The first thing we must do is show that $|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, which is known as Schwarz's inequality. Without this we cannot be permitted to always relate $\dfrac{\mathbf{x}\cdot\mathbf{b}}{|\mathbf{x}||\mathbf{b}|}$ to θ via inverse trigonometric functions. These details will occur in chapter 6 where we find that by using the inner-product on vectors from $\mathbb{R}^n$ we will define the notion of angle and from that distance. Using these definitions and Schwarz's inequality will then give us a triangle-inequality for arbitrary finite-dimensional vectors. This is to say that the algebra of vectors in $\mathbb{R}^n$ carries its own definition of angle and length - very nice of it don't you think? Also, it should be noted that these results exist for certain so-called infinite-dimensional spaces but are harder to prove and that the study of linear transformation of such spaces is the general setting for quantum mechanics - see MATH503:Functional Analysis for more details.

[7]To see why this is true d, differentiate an arbitrary element of $\mathbf{AB}$ to find $\dfrac{d}{d\theta}[\mathbf{AB}]_{ij} = \dfrac{d}{d\theta}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n}\dfrac{da_{ik}}{d\theta}b_{kj} + a_{ik}\dfrac{db_{kj}}{d\theta}$

Define the *commutator* and *anti-commutator* of two square matrices to be,

$$[\cdot,\cdot] : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \text{ such that } [\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \quad \text{for all } \mathbf{A},\mathbf{B} \in \mathbb{C}^{n\times n},$$
$$\{\cdot,\cdot\} : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, \text{ such that } \{\mathbf{A},\mathbf{B}\} = \mathbf{AB} + \mathbf{BA}, \quad \text{for all } \mathbf{A},\mathbf{B} \in \mathbb{C}^{n\times n},$$

respectively. Also define the *Kronecker delta* and *Levi-Civita* symbols to be,

$$\delta_{ij} : \mathbb{N}\times\mathbb{N} \to \{0,1\}, \text{ such that } \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

$$\epsilon_{ijk} : (i,j,k) \to \{-1,0,1\}, \text{ such that } \epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1) \text{ or } (3,1,2), \\ -1, & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2) \text{ or } (2,1,3), \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

respectively. Also define the so-called *Pauli spin-matrices* (PSM) to be,

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5.1. **The PSM are *self-adjoint* matrices.** Show that $\sigma_m = \sigma_m^{\text{H}}$ for $m = 1,2,3$.

5.2. **The PSM are *unitary* matrices.** Show that $\sigma_m\sigma_m^{\text{H}} = \mathbf{I}$ for $m = 1,2,3$ where $[\mathbf{I}]_{ij} = \delta_{ij}$.

5.3. **Trace and Determinant.** Show that $\operatorname{tr}(\sigma_m) = 0$ and $\det(\sigma_m) = -1$ for $m = 1,2,3$.

5.4. **Anti-Commutation Relations.** Show that $\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1,2,3$ and $j = 1,2,3$.

5.5. **Commutation Relations.** Show that $[\sigma_i,\sigma_j] = 2\sqrt{-1}\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$ for $i = 1,2,3$ and $j = 1,2,3$.

5.6. **Spin$-\frac{1}{2}$ Systems.** In quantum mechanics spin one-half particles, typically electrons[8] , have 'spins' characterized by the following vectors:

$$\mathbf{e}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{e}_u$ represents spin-up and $\mathbf{e}_d$ represents spin-down.[9] The following matrices,

$$\mathbf{S}_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S}_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

are linear transformations, which act on $\mathbf{e}_u$ and $\mathbf{e}_d$.

5.6.1. *Projections of Spin$-\frac{1}{2}$ Systems.* Compute and describe the effect of the transformations, $\mathbf{S}_+(\mathbf{e}_u + \mathbf{e}_d)$, and $\mathbf{S}_-(\mathbf{e}_u + \mathbf{e}_d)$.

5.6.2. *Properties of Projections.* $\mathbf{S}_+$ and $\mathbf{S}_-$ are projection transformations. Projection transformations are known to destroy information. Justify this in the case of $\mathbf{S}_+$ by showing that $\mathbf{S}_+$ is neither one-to-one nor onto $\mathbb{R}^2$.

[8] In general these particles are called fermions. `http://en.wikipedia.org/wiki/Spin-1/2`, `http://en.wikipedia.org/wiki/Fermions`

[9] In quantum mechanics, the concept of spin was originally considered to be the rotation of an elementary particle about its own axis and thus was considered analogous to classical angular momentum subject to quantum quantization. However, this analogue is only correct in the sense that spin obeys the same rules as quantized angular momentum. In 'reality' spin is an intrinsic property of elementary particles and it is the roll of quantum mechanics to understand how to associate quantized particles with spin to their associated background field in such a way that certain field properties/symmetries are preserved. This is studied in so-called quantum field theory. `http://www.physics.thetangentbundle.net/wiki/Quantum_mechanics/spin`, `http://en.wikipedia.org/wiki/Spin_(physics)`, `http://en.wikipedia.org/wiki/Quantum_field_theory`