E. Kreyszig, Advanced Engineering Mathematics, $8^{\text {th }}$ ed.

## Lecture: Review of Functions Module: 03

Suggested Problem Set: Suggested Problems : \{null \}

| Quote of Review |
| :--- | | Everybody's talking about the stormy weather and what's a man do to but work out |
| :--- |
| whether it's true? Looking for a man with a focus and a temper who can open up a map |
| and see between one and two |

## 1 Introduction

In the following I provide a non-technical review of the seemingly innocuous concept of a function. The goal of this 'review' is to provide the background needed for a discussion on Fourier series. It is assumed that the reader has seen at least some of this material, maybe only in passing, and will not be shocked by the notion of a Taylor series or concepts like periodicity and continuity. The review will conclude with material it is assumed the reader has never seen and is presented to introduce the reader to some special functions, which are likely to appear when we survey PDE's. ${ }^{1}$

## 2 Review

In the following we assume a set is a collection of objects, which can be well-defined. ${ }^{2}$ In everything we do we will assume the set to be some, not necessarily proper, subset of $\mathbb{R}^{n}$ and that the objects are the vectors from linear algebra. ${ }^{3}$ In this context we define functions to map between sets.

Definition: A function $f$ is a rule or mapping, which assigns to elements from its domain, $D$, unique elements in its range, $R$, and we write ${ }^{4}$

$$
\begin{align*}
f & : \quad D \rightarrow R .  \tag{1}\\
f(x) & =\quad y, \quad x \in D, y \in R . \tag{2}
\end{align*}
$$

Along with this concept we have the following vocabulary:

- One-to-one : We say that the function $f$ is one-to-one, or injective, if for all $a, b \in D$, and if $f(a)=f(b)$ then $a=b$. The classical examples are $f_{1}(x)=x^{2}$ and $f_{2}(x)=x^{3}$. For $f_{1}$ we have $f_{1}(-1)=f_{1}(1)$ but $1 \neq-1$. Thus, $f_{1}$ is not injective, while a quick calculation shows that $f_{2}$ is.
- Onto : We say that a function $f$ is onto, or surjective, if for all $y \in R$ there exists an element, $x \in D$ such that $f(x)=y$. That is, all elements of $R$ are values of $f$ with respect to its argument, $x$. As an example we note that the mapping $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x)=\sin (x)$ is not onto since not all $y \in \mathbb{R}$ cannot be represented by $f(x)=y$.

[^0]- Inverse Function : If a function is one-to-one and onto its range then the function is called bijective and thus its inverse function $f^{-1}$ exists and is a bijection. For example if we consider $f_{1}$ from above then we see that it is neither one-to-one with $\mathbb{R}$ nor onto $\mathbb{R}$. If we restrict $f_{1}$ to have domain $[0, \infty)$ and consider it as a mapping onto $\mathbb{R}^{+} \cup\{0\}$, then we have the inverse function $f_{1}^{-1}(y)=\sqrt{y}=x$. ${ }^{5}$
- Periodic Function : We say that a function, $f$, is periodic if for some $p \in \mathbb{R}^{+}$we have that $f(x+p)=$ $f(x)$ for all $x \in D$. The prototypical examples are $f_{3}(x)=\cos (x)$ and $f_{4}(x)=\sin (x)$, which both have period $2 \pi .{ }^{6}$ It is also interesting to note that if $f$ is $p$-periodic then for any integer $n$ we have that $f(x+n p)=f(x)$ for all $x \in D .{ }^{7}{ }^{8}$ We also have that the sum of two periodic functions of period $p$ also has period $p .{ }^{9}$
- Continuous Function : Quite formally we say that the function $f: D \rightarrow R$ where $D, R \subset \mathbb{R}$ is said to be continuous at $c \in D$ if for any $\epsilon>0$ there exists $\delta>0$ such that for all $x \in(c-\delta, c+\delta)$ we have that $\|x-c\|<\delta$ implies that $\|f(x)-f(c)\|<\epsilon$. We take this to mean that if a function is continuous then no matter how small the 'neighborhood' of $f(c)$ is we can choose a neighborhood of $x$ 's about $c$ so that $f$ maps the neighborhood about $c$ to the neighborhood about $f(c)$. More intuitively, we take this to mean that the graph of a continuous function cannot have 'holes' or 'jumps.' As one might expect, this is a powerful concept used to prove important properties of functions such as the intermediate and mean value theorems. ${ }^{10}$ It is interesting to note that while continuity at a point is a consequence of differentiability the converse need not be true. ${ }^{11}$ Maybe more surprising is that all continuous functions are integrable. So, continuity implies integrability but not necessarily differentiability. ${ }^{12}$
Functions are often classified based on whether they satisfy some extra structures. For instance, one often considers the space of continuous functions, $C((a, b))$ whose domain is the open line segment $I=(a, b), a, b \in$ $\mathbb{R}, a \leq b$ or the space of continuous functions, which have $k$-many continuous derivatives on $I, C^{k}(I)$. Frequently, we take $k \rightarrow \infty$ and talk about a function through its Taylor series: ${ }^{13}$

$$
\begin{equation*}
f(x)=\sum_{n=0}^{\infty} \frac{f^{n}\left(x_{0}\right)}{n!}\left(x-x_{0}\right)^{n} . \tag{3}
\end{equation*}
$$

This is a powerful representation and allows us to formally derive the beautiful formula,

$$
\begin{equation*}
e^{i x}=\cos (x)+i \sin (x), \tag{4}
\end{equation*}
$$

due to Euler and bears his name. Not only does this formula make apparent the periodic dynamics of a simple harmonic oscillator, $m y^{\prime \prime}+k y=0$, but serves to connect the three most important functions studied in transcendental calculus. Though these functions are 'found' many ways the most straightforward arguments tend to be:

- Define the function that, upon a single differentiation, returns itself.
- Define the function that, upon two differentiations, returns the negative of itself.

There is no simple reason that these two arguments should find connection through the imaginary number system. However, it is clear by the derivation that algebraically their structures are the same. While, these

[^1]are impressive functions, indeed, they are not the only functions expressible by linear combinations of power functions.

## 3 Bessel's Function and the Gamma Function

There are other functions, which are expressed as infinite series and have been given considerable attention in the last 100 years. These functions are typically called special functions. For example we consider the ODE,

$$
\begin{equation*}
x^{2} y^{\prime \prime}+x y^{\prime}+\left(x^{2}-\nu^{2}\right) y=0 \tag{5}
\end{equation*}
$$

commonly called Bessel's equation of order $\nu$. Assuming that the unknown solution has the form,

$$
\begin{equation*}
y(x)=\sum_{m=0}^{\infty} a_{m} x^{m+r}, \quad r \in \mathbb{R} \tag{6}
\end{equation*}
$$

defines a recurrence relation and general formula for the coefficients as,

$$
\begin{equation*}
a_{2 m}=\frac{(-1)^{m} a_{0}}{2^{2 m} m!(\nu+1)(\nu+2) \cdots(\nu+m)} . \tag{7}
\end{equation*}
$$

In the case that $\nu$ is an integer then the denominator of (7) becomes a factorial function and we define Bessel's function of the first kind of order $n$ as,

$$
\begin{equation*}
J_{n}(x)=x^{n} \sum_{m=0}^{\infty} \frac{(-1)^{m} x^{2 m}}{2^{2 m+n} m!(n+m)!} . \tag{8}
\end{equation*}
$$

Bessel's function is an important special function commonly studied in physics and applied mathematics. Typically, the function defines the 'spatial modes' of the time evolution of Laplace's equation in cylindrical and spherical coordinates and thus appears heavily in wave propagation, diffusive flows and potential theory. It is common to have $\nu$ appear as an integer but it often appear as a half-integer. In this case the denominator of (7) cannot be written as a factorial function, since the factorial is defined as a function on the counting numbers, $\mathbb{N}$. To work around this we must abstract the factorial function to the decimal number system. In doing so we arrive at the so-called Gamma function,

$$
\begin{equation*}
\Gamma(\nu)=\int_{0}^{\infty} e^{-t} t^{\nu-1} d t \tag{9}
\end{equation*}
$$

Using integration by parts one can show that,

$$
\begin{equation*}
\Gamma(\nu+1)=\nu \Gamma(\nu) \tag{10}
\end{equation*}
$$

and thus, by repeated application and noticing that $\Gamma(1)=1$, we recover the factorial function for $\nu=n \in \mathbb{N}$

$$
\begin{equation*}
\Gamma(n+1)=n \Gamma(n)=n(n-1) \Gamma(n-1)=n(n-1)(n-2) \Gamma(n-2)=n(n-1)(n-2) \cdots \Gamma(1)=n! \tag{11}
\end{equation*}
$$

However, it is also the case that for $\nu=3 / 2$, which is clearly not an integer, the statement,

$$
\begin{equation*}
\Gamma\left(\frac{3}{2}\right)=\int_{0}^{\infty} e^{-t} \sqrt{t} d t \tag{12}
\end{equation*}
$$

makes 'sense' and allows us to extend the factorial function to non-integer argument. Doing so, allows us to write the solution to (5) as,

$$
\begin{equation*}
J_{\nu}(x)=x^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^{m} x^{2 m}}{2^{2 m+\nu} m!\Gamma(\nu+1+m)!} \tag{13}
\end{equation*}
$$

for non-negative $\nu$.


[^0]:    ${ }^{1}$ Have you ever thought about trying to write down an expression for a function, which oscillates but is not periodic? Or have you wanted to calculate something like the factorial of a decimal number? Well, some have and did!
    ${ }^{2}$ In this sense we use a naive set theory so as to avoid some natural weirdness dealt with in the axiomatic set theory. The classic example called Russell's paradox and arises within naive set theory by considering the set of all sets that are not members of themselves. Such a set appears to be a member of itself if and only if it is not a member of itself, hence the paradox. Perhaps a more entertaining version is the Barber paradox, which states, that there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber. It seems reasonable to imagine that the barber obeys the following rule: He shaves only those men who do not shave themselves. We then ask, does the barber shave himself?
    ${ }^{3}$ It will often be the case that $n=1$, but when we get to PDE we will work in $n=3$.
    ${ }^{4}$ This is to say that a function can not take one object to many objects.

[^1]:    ${ }^{5}$ Inverse functions also have 'nice' composition properties in that the composition of two bijections is itself a bijection. However, in the converse we have that if the composition of two functions is a bijection then the most one can say is that one function is injective and one is surjective.
    ${ }^{6}$ Periodic functions can, of course, be more complicated. The needn't be $2 \pi$-periodic nor sinusoids. For example consider $g(x)=\operatorname{sign}(\sin (x))=\left\{\begin{array}{cc}-1, & x \in(-\pi, 0) \\ 0, & x=0 \\ 1, & x \in(0, \pi)\end{array}\right.$ and signal processing.
    ${ }^{7}$ See the argument in the text page 478.
    ${ }^{8}$ From this we can conclude that for some integer, $m, g(x)=\sin (m x)$ is $2 \pi / m$-periodic function but also a $2 \pi$ periodic function.
    ${ }^{9}$ This allows us to say that $f(x)=\sum_{n=0}^{\infty} a_{n} \cos (n x)+b_{n} \sin (n x)$ is a $2 \pi$-periodic function.
    ${ }^{10}$ We also have that the finite sum of continuous functions is continuous as are compositions of continuous functions.
    ${ }^{11}$ Recall cusp points are points of a continuous function where the derivative fails to be unique and thus ill-defined.
    ${ }^{12}$ For this reason Fourier series tend to be more versatile than Taylor series.
    ${ }^{13}$ In this case centered about $x_{0}$.

