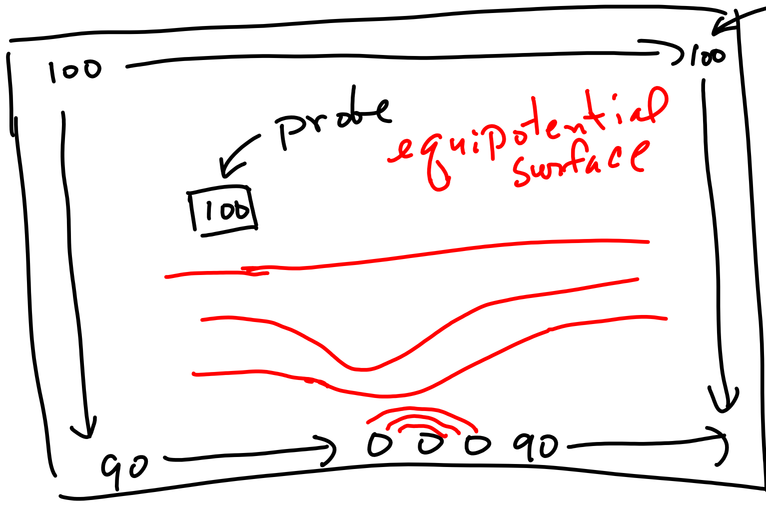
Hmwk soln assign 8

1.)

boundary values



The electric field is perpendicular to the equipotential surfaces. Ohm's law says J = sigma E so current flows perpendicular to the equipotential surfaces. Note that there is a large E and J when the probe is over the Titanic. If there is no Titanic, the number of equipot. surfaces is much smaller: 100-90 vs 100-0 so much less current flows.

2.) Maxwell's eqns in vaccum:

$$\vec\nabla \cdot \vec B = 0 \qquad \vec\nabla \times \vec B = \overbrace{\mu_0 \epsilon_0 \frac{\partial \vec E}{\partial t}}^{\vec J_{displacement}}$$

$$\vec\nabla \cdot \vec E = 0 \qquad \vec\nabla \times \vec E = -\frac{\partial \vec B}{\partial t}$$

Use the vector identity

$$\vec\nabla \times \vec\nabla \times \vec V = \vec\nabla(\vec\nabla \cdot \vec V) - \nabla^2 \vec V$$

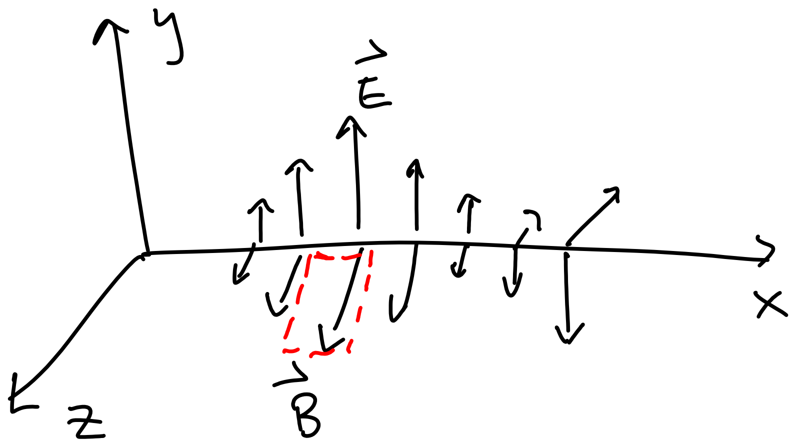
$$\vec\nabla \times \vec\nabla \times \vec B = \vec\nabla(\underbrace{\vec\nabla \cdot \vec B}_{=0}) - \nabla^2 \vec B = \mu_0 \epsilon_0 \frac{\partial (\overbrace{\vec\nabla \times \vec E}^{= -\frac{\partial \vec B}{\partial t}})}{\partial t}$$

$$\nabla^2 \vec B - \mu_0 \epsilon_0 \frac{\partial^2 \vec B}{\partial t^2} = 0$$

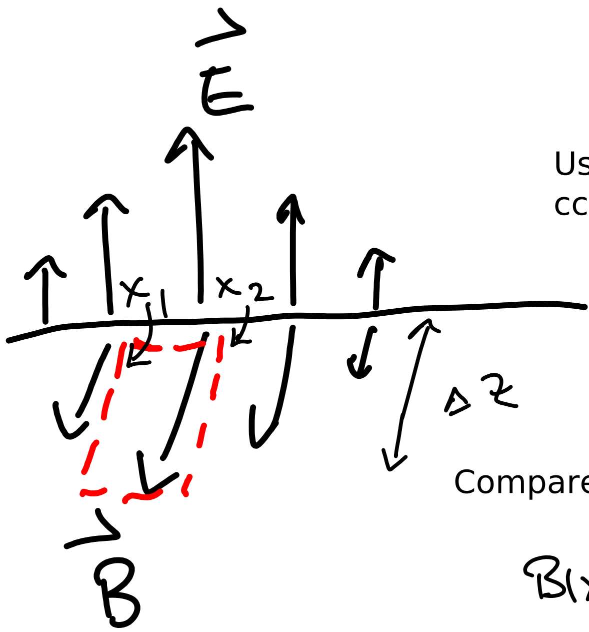
$$\vec\nabla \times \vec\nabla \times \vec E = \vec\nabla(\underbrace{\vec\nabla \cdot \vec E}_{=0}) - \nabla^2 \vec E = -\frac{\partial}{\partial t} \overbrace{\vec\nabla \times \vec B}^{= \mu_0 \epsilon_0 \frac{\partial \vec E}{\partial t}} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec E}{\partial t^2}$$

$$\nabla^2 \vec E - \mu_0 \epsilon_0 \frac{\partial^2 \vec E}{\partial t^2} = 0$$

3.)



$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \xrightarrow[\text{theorem}]{\text{Stokes}} \oint \vec{B}\cdot d\vec{r} = \mu_0\epsilon_0\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{a}$$



Using the right hand rule, with E up, integrate ccw as viewed down the y-axes.

$$\oint \vec{B}\cdot d\vec{r} = -B(x_2)\Delta z + B(x_1)\Delta z$$

Compare the path direction here with the one in lecture.

$$B(x_2) = B(x_1) + \frac{\partial B}{\partial x}\Delta x$$

$$LHS = -B(x_2)\Delta z + B(x_1)\Delta z = -\left(B(x_1) + \frac{\partial B}{\partial x}\Delta x\right)\Delta z + B(x_1)\Delta z = -\frac{\partial B}{\partial x}\Delta x\Delta z$$

$$RHS = \mu_0\epsilon_0\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{a} \longrightarrow \mu_0\epsilon_0\frac{\partial E}{\partial t}\Delta x\Delta z$$

A $\boxed{-\frac{\partial B}{\partial x} = \mu_0\epsilon_0\frac{\partial E}{\partial t}}$

From class we had

$$\text{B} \quad \boxed{\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}} \qquad \text{2 coupled PDE's}$$

Diff A wrt x & B wrt t $\Rightarrow$
$$\begin{cases} -\dfrac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \dfrac{\partial^2 E}{\partial x \partial t} \\ \dfrac{\partial^2 E}{\partial t \partial x} = -\dfrac{\partial^2 B}{\partial t^2} \end{cases}$$

$$-c^2 \frac{\partial^2 B}{\partial x^2} = -\frac{\partial^2 B}{\partial t^2} \qquad \text{the wave eqn!}$$

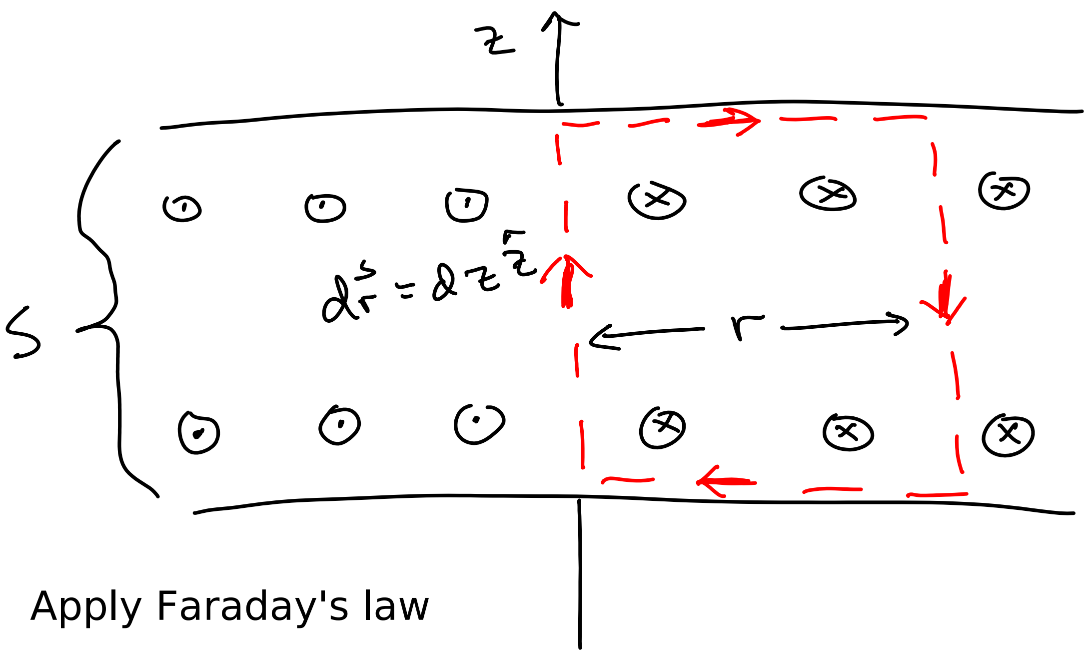
4.) The perturbation series is given by

$$E_{tot}(t) = E_1 e^{i\omega t} + E_2(t) + \cdots$$
$$B_{tot} = B_1 + B_2 + \cdots$$

In the March 19 lecture we assume a uniform electric field $E_1 e^{i\omega t}$

and then calculated

$$B_1 = \frac{\mu_0 \epsilon_0}{2\pi a^2} r E_1 i\omega e^{i\omega t}$$



let E be positive in upward direction ↑

Apply Faraday's law

$$\int \vec{E}_2 \cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}_1 \cdot d\vec{a}$$

2nd iteration for E_2

$$-E_2 s = -\frac{\partial}{\partial t}\int \frac{i\omega}{2}\frac{r}{c^2} E_0 e^{i\omega t}\, s\, dr$$

$$E_2 = -\frac{\omega^2 r^2}{4}\frac{1}{c^2} E_0 e^{i\omega t}$$

2nd iteration for B_2

$$\oint \vec{B}_2 \cdot d\vec{r} = \mu_0 \int \epsilon_0 \frac{\partial \vec{E}_2}{\partial t}\, 2\pi r\, dr$$

$$B_2 = -\frac{i\omega^3 r^3}{16 c^4} E_0 e^{i\omega t}$$

3rd

$$\oint \vec{E}_3 \cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}_2 \cdot d\vec{a}$$

$$E_3 s = -\frac{\partial}{\partial t}\int -\frac{i\omega^3 r^3}{16 c^4} E_0 e^{i\omega t}\, s\, dr \Rightarrow E_3 = \frac{\omega^4 r^4}{64 c^4} E_0 e^{i\omega t}$$

5.) For homework show $$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \epsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$$

substitute

us $$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\vec{E}\cdot\vec{J} = \vec{E}\cdot\left(\frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \epsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

For homework show

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Look up and use the vector identity for $\vec{\nabla}\cdot(\vec{E}\times\vec{B})$.

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot\underbrace{\vec{\nabla}\times\vec{E}}_{= -\frac{\partial B}{\partial t}} - \vec{E}\cdot\vec{\nabla}\times\vec{B}$$

$-\frac{\partial B}{\partial t}$ from Maxwell's eqns

So $$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

For homework use $\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(B^2)$ & $\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(E^2)$

to show that

$$\vec{E}\cdot\vec{J}=-\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2+\frac{1}{\mu_0}B^2\right)-\frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Start with the result from above

$$\vec{E}\cdot\vec{J}=\frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B})-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

But we also showed

$$-\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Next use

$$\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(B^2) \quad \& \quad \vec{E}\cdot\frac{\partial\vec{E}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

To get

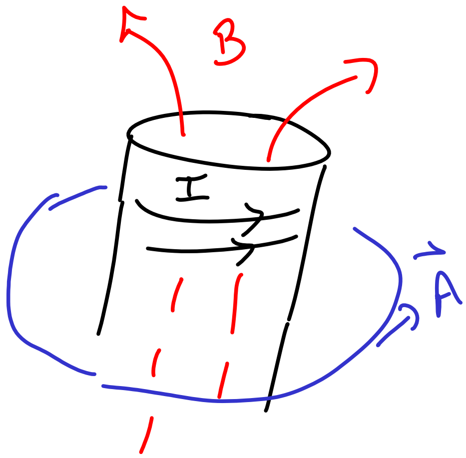
$$\vec{E}\cdot\vec{J}=-\frac{\partial}{\partial t}\frac{1}{2}\left(\epsilon_0 E^2+\frac{B^2}{\mu_0}\right)-\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Integrate over some closed volume and apply the divergence theorem to get

$$\frac{dW}{dt}=-\frac{d}{dt}\underbrace{\int\frac{1}{2}\left(\epsilon_0 E^2+\frac{1}{\mu_0}B^2\right)d\tau}_{u_{EM}}-\frac{1}{\mu_0}\oint\underbrace{(\vec{E}\times\vec{B})}_{\vec{S}}\cdot d\vec{a}$$

$\vec{S}=\frac{1}{\mu_0}\vec{E}\times\vec{B}$ is Poynting vector

6.)



$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

stokes th.

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a}$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\int \vec{\nabla}\times\vec{A}\cdot da$$

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\oint \vec{A}\cdot d\vec{r}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t}$$

Impulse = $\vec{F} = \Delta\vec{p}/\Delta t$

$$\vec{F} = q\vec{E} = -q\frac{\partial\vec{A}}{\partial t} \approx \frac{\Delta\vec{p}}{\Delta t} \Rightarrow \Delta\vec{p} = -q\Delta\vec{A}$$

$$\vec{p} = -q\vec{A}$$