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## HOMEWORK 6

$$\text{[1] GIVEN, } ay'' + by' + cy = f(x) \quad (1)$$

(a) FIND ALL SOLUTIONS TO THE HOMOGENEOUS VERSION OF (1).

ASSUME:  $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$

$ar^2 + br + c = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{WHERE } D = b^2 - 4ac$

D	$y_h(x)$
$> 0$	$= c_1 e^{\left(\frac{-b + \sqrt{D}}{2a}\right)x} + c_2 e^{\left(\frac{-b - \sqrt{D}}{2a}\right)x}$
$= 0$	$= c_1 e^{\left(\frac{-b}{2a}\right)x} + c_2 x e^{\left(\frac{-b}{2a}\right)x}$
$< 0$	$= e^{\left(\frac{-b}{2a}\right)x} \left[ c_1 \cos\left(\frac{\sqrt{ D }}{2a}x\right) + c_2 \sin\left(\frac{\sqrt{ D }}{2a}x\right) \right]$

(b) FILL IN THE FOLLOWING TABLE WITH THE CHOICES YOU WOULD MAKE FOR THE PARTICULAR SOLUTION.

$f(x)$	$y_p(x)$
$x^2$	$Ax^3 + Bx^2 + Cx + Dx + E$
$\cos(\beta x)$	$A \cos(\beta x) + B \sin(\beta x)$
$e^{\alpha x} + \sin(x) + x$	$Ae^{\alpha x} + B \cos(x) + C \sin(x) + Dx + E$
$e^{\alpha x} \sin(\beta x)$	$Ae^{\alpha x} \cos(\beta x) + Be^{\alpha x} \sin(\beta x)$
$x^2 e^{\alpha x} \cos(\beta x)$	$(Ax^2 + Bx + C)e^{\alpha x} \cos(\beta x) + (Dx^2 + Ex + F)e^{\alpha x} \sin(\beta x)$

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(c) ASSUME THAT  $a=1$ ,  $b=0$ ,  $c=9$  AND  $f(x) = \cos(3x)$ . FIND THE GENERAL SOLUTION TO (1)

$$D = b^2 - 4ac = -36$$

FROM (a) WE HAVE THAT  $y_c = C_1 \cos(3x) + C_2 \sin(3x)$

$$y_p = Ax \cos(3x) + Bx \sin(3x)$$

$$y_p' = -3Ax \sin(3x) + A \cos(3x) + 3Bx \cos(3x) + B \sin(3x)$$

$$\begin{aligned} y_p'' &= -9Ax \cos(3x) - 3A \sin(3x) - 3A \sin(3x) - 9Bx \sin(3x) + 3B \cos(3x) + 3B \cos(3x) \\ &= -9Ax \cos(3x) - 6A \sin(3x) - 9Bx \sin(3x) + 6B \cos(3x) \end{aligned}$$

$$y_p'' + 9y_p =$$

$$\begin{aligned} -9Ax \cos(3x) - 6A \sin(3x) - 9Bx \sin(3x) + 6B \cos(3x) + 9Ax \cos(3x) + 9Bx \sin(3x) \\ = \cos(3x) \end{aligned}$$

$$\Rightarrow -6A \sin(3x) + 6B \cos(3x) = \cos(3x) \Rightarrow \begin{aligned} A &= 0 \\ B &= \frac{1}{6} \end{aligned}$$

$$y_p = \frac{1}{6} x \sin(3x)$$

$$y_c + y_p = \boxed{y = C_1 \cos(3x) + C_2 \sin(3x) + \frac{x}{6} \sin(3x)}$$

(2) CONSIDER THE BOUNDARY VALUE PROBLEM:

$$y'' + \lambda y = 0 \quad (2)$$

$$y'(0) = y'(1) = 0 \quad (3)$$

(a) ASSUMING  $\lambda \in \mathbb{R}$  FIND THE GENERAL SOLUTION TO (2)

ASSUME:  $y = e^{rx}$

$$r^2 e^{rx} + \lambda e^{rx} = 0$$

$$y' = r e^{rx}$$

$$r^2 + \lambda = 0$$

$$y'' = r^2 e^{rx}$$

$$r = \pm \sqrt{-\lambda}$$

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$\lambda$	$y(x) =$
$< 0$	$C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} = C_1 \cosh(\sqrt{-\lambda}x) + C_2 \sinh(\sqrt{-\lambda}x)$
$= 0$	$C_1 + C_2 x$
$> 0$	$C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

(b) APPLY THE BOUNDARY CONDITIONS (3) AND CALCULATE THE POSSIBLE VALUES OF  $\lambda$ .

CASE 1:  $\lambda < 0$ 

$$y'(x) = [C_1 \sinh(\sqrt{-\lambda}x) + C_2 \cosh(\sqrt{-\lambda}x)] \sqrt{-\lambda}$$

$$y'(0) = 0 = C_2 \cosh(0) \Rightarrow C_2 = 0$$

$$y'(1) = 0 = C_1 \sinh(\sqrt{-\lambda}) + C_2 \cosh(\sqrt{-\lambda})$$

BECAUSE  $\lambda \neq 0$ ,  $C_1 = 0$  AND  $y(x) = 0$ , THE TRIVIAL SOLUTION

CASE 2:  $\lambda = 0$ 

$$y'(x) = C_2$$

$$y'(0) = 0 = C_2$$

$$y'(1) = 0 = C_2$$

$C_2 = 0$ ,  $C_1 \in \mathbb{R}$  THEREFORE

$y(x) = C_1$ , A NON-TRIVIAL SOLUTION

AND 0 IS A POSSIBLE VALUE OF  $\lambda$

CASE 3:  $\lambda > 0$ 

$$y'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda}x)$$

$$y'(0) = 0 = C_2 \sqrt{\lambda} \Rightarrow \text{BECAUSE } \lambda \neq 0, C_2 = 0$$

$$y'(1) = 0 = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda})$$

ASSUME  $C_1 \neq 0$

$$\lambda = n^2 \pi^2 \quad n = 1, 2, 3, \dots$$

$$y(x) = C_1 \cos(\sqrt{\lambda}x)$$

THE POSSIBLE VALUES OF  $\lambda$  ARE

$$\lambda = 0$$

$$\lambda = n^2 \pi^2 \quad n = 1, 2, 3, \dots$$

AND THE GENERAL SOLUTION IS:

$$y(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\pi x)$$

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3) CONSIDER THE ORDINARY DIFFERENTIAL EQUATION:

$$y'' - y = 0 \quad (4)$$

(a) SHOW THAT  $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$  IS A SOLUTION TO (4).

$$y(x) = b_1 \sinh(x) + b_2 \cosh(x)$$

$$y'(x) = b_1 \cosh(x) + b_2 \sinh(x)$$

$$y''(x) = b_1 \sinh(x) + b_2 \cosh(x)$$

$$y'' - y = b_1 \sinh(x) + b_2 \cosh(x) - (b_1 \sinh(x) + b_2 \cosh(x)) = 0$$

(b) SHOW THAT IF  $c_1 = \frac{b_1 + b_2}{2}$  AND  $c_2 = \frac{b_1 - b_2}{2}$  THEN

$$y(x) = c_1 e^x + c_2 e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$$

$$y(x) = c_1 e^x + c_2 e^{-x}$$

$$= \left(\frac{b_1 + b_2}{2}\right) e^x + \left(\frac{b_1 - b_2}{2}\right) e^{-x}$$

$$= \frac{b_1 (e^x + e^{-x})}{2} + \frac{b_2 (e^x - e^{-x})}{2}$$

$$= b_1 \cosh(x) + b_2 \sinh(x)$$

THE HYPERBOLIC SINE AND COSINE HAVE THE FOLLOWING TAYLOR SERIES REPRESENTATIONS:

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad (5)$$

(c) ASSUME THAT  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  AND FIND THE GENERAL SOLUTION OF (4)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

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$$y'' - y = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$\uparrow$   $\uparrow$   
 $k=n-2$   $k=n$

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} [a_{k+2} (k+2)(k+1) - a_k] x^k = 0$$

SINCE  $x^k$  CAN'T BE 0, WE ASSUME

$$a_{k+2} (k+2)(k+1) - a_k = 0$$

$$\Rightarrow a_{k+2} = \frac{a_k}{(k+2)(k+1)} \quad k = 0, 1, 2, 3, \dots$$

$$k=0 \quad a_2 = \frac{a_0}{2 \cdot 1}$$

$$k=1 \quad a_3 = \frac{a_1}{3 \cdot 2 \cdot 1}$$

$$k=2 \quad a_4 = \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$k=3 \quad a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$k=4 \quad a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$k=5 \quad a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

FOR EVEN  $k$ :  $a_{2n} = \frac{a_0}{(2n)!} \quad n = 0, 1, 2, 3, \dots$

FOR ODD  $k$ :  $a_{2n+1} = \frac{a_1}{(2n+1)!} \quad n = 0, 1, 2, 3, \dots$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{(2n+1)}}{(2n+1)!}$$

$$y(x) = a_0 \cosh(x) + a_1 \sinh(x)$$

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- 4 (a) EXPLAIN THE PHYSICAL INTERPRETATION OF THE MAXIMUM PRINCIPLE.

TEMPERATURE COMES FROM EITHER A SOURCE OR FROM EARLIER IN TIME, BUT IS NOT CREATED FROM NOTHING

- (b) EXPLAIN THE RELATIONSHIP BETWEEN THE HEAT EQUATION AND THE DIFFUSION EQUATION.

THE DIFFUSION EQUATION IS A NON-LINEAR FORM OF THE HEAT EQUATION THAT IS DERIVED FROM THE CONTINUITY EQUATION USING FICK'S FIRST LAW

- (c) WRITE DOWN THE LINEAR WAVE EQUATION AND EXPLAIN  $c$  AND  $c(u)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{WHERE } c \text{ IS THE PROPAGATION SPEED OF THE WAVE. } c(u) \text{ IS A WAVE SPEED DEPENDENT ON THE AMPLITUDE.}$$

- (d) WHAT PHYSICAL PHENOMENON MODELED BY THE LINEAR WAVE EQUATION? THE NON-LINEAR WAVE EQUATION?

LINEAR WAVE EQUATION MODELS WAVES } VIBRATIONS. THE NON-LINEAR WAVE EQUATION PROVIDES A MORE REALISTIC MODEL

- (e) DEFINE DISPERSION AND GIVE A PHYSICAL EXAMPLE.

DISPERSION IS WHEN THE VELOCITY OF A WAVE IS A FUNCTION OF FREQUENCY, SUCH AS SOUND WAVES TRAVELING THROUGH WATER.

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5) SHOW THAT THE FOLLOWING FUNCTIONS ARE SOLUTIONS TO THEIR CORRESPONDING P.D.E'S

(a)  $u(x,t) = f(x-ct) + g(x+ct)$  ; 1-D WAVE EQUATION

$$\frac{\partial u}{\partial t} = -cf' + cg' \quad \leftarrow \text{CHAIN RULE} \quad \frac{\partial u}{\partial x} = f' + g'$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 f'' + c^2 g'' \quad \frac{\partial^2 u}{\partial x^2} = f'' + g''$$

$$\text{1-D WAVE EQUATION} \rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow c^2(f'' + g'') = c^2(f'' + g'')$$

(b)  $u(x,t) = e^{-4w^2 t} \sin(wx)$  ; 1-D HEAT EQUATION,  $c=2$

$$\frac{\partial u}{\partial t} = -4w^2 e^{-4w^2 t} \sin(wx) \quad \frac{\partial u}{\partial x} = w e^{-4w^2 t} \cos(wx)$$

$$\frac{\partial^2 u}{\partial x^2} = -w^2 e^{-4w^2 t} \sin(wx)$$

$$\text{1-D HEAT EQUATION} \rightarrow \frac{\partial u}{\partial t} = (c^2) \frac{\partial^2 u}{\partial x^2}$$

$c=2$

$$\Rightarrow -4w^2 e^{-4w^2 t} \sin(wx) = -4w^2 e^{-4w^2 t} \sin(wx)$$

(c)  $u(x,y) = x^4 + y^4$  ; 2-D POISSON EQUATION;  $f(x,y) = 12(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = 4x^3$$

$$\frac{\partial u}{\partial y} = 4y^3$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2$$

$$\frac{\partial^2 u}{\partial y^2} = 12y^2$$

$$\text{2-D POISSON EQUATION} \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) = 12(x^2 + y^2)$$

$f(x,y) = 12(x^2 + y^2)$

$$\Rightarrow 12(x^2 + y^2) = 12(x^2 + y^2)$$

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$$(d) u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} ; \text{ 3-D LAPLACE EQUATION}$$

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \quad \leftarrow \text{PRODUCT RULE}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-1}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-1}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\text{3-D LAPLACE EQUATION} \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\Rightarrow \frac{3x^2 + 3y^2 + 3z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} = 0$$

$$\Rightarrow \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} = 0$$

$$\Rightarrow \frac{3}{(x^2 + y^2 + z^2)^{3/2}} + \frac{-3}{(x^2 + y^2 + z^2)^{3/2}} = 0$$

$$\Rightarrow 0 = 0$$