Problem 2:

Since

$$U_x(0,t) = U_x(L,t) = 0$$

We assume

$$U(x,t) = a_0 G_0(t) + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right) \cdot G_n(t)$$

Note:
call
$a_0 G_0 = G_0(t)$
$a_n G_n(t) = G_n(t)$

Where G_n, G_0 are unknown dynamics for a Fourier cosine series. For this reason we assume

$$F(x,t) = \cancel{a}\, f_0 + \sum_{n=1}^{\infty} f_n \cos\left(\frac{n\pi}{L} x\right)$$

Where

$$f_0 = \frac{1}{L}\int_0^L F(x,t)\,dx, \quad f_n = \frac{2}{L}\int_0^L F(x,t)\cos\left(\frac{n\pi}{L} x\right)$$

In this case the heat eqn becomes,

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} - F(x,t) = G_0'(t) - f_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L} x\right)\left[G_n' + \left(\frac{cn\pi}{L}\right)^2 G_n - f_n\right] =$$

By orthogonality of $\{1, \cos(\pi x/L), \cos(\frac{2\pi}{L}x), \dots\}$

We have

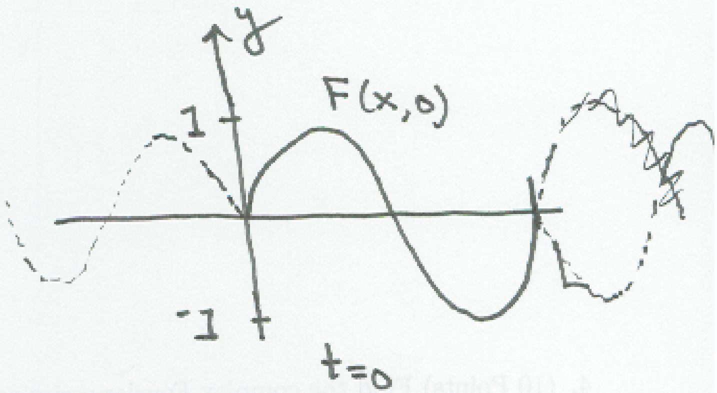
$$(**)\quad G_n' + \alpha_n G_n = f_n, \quad n = 1, 2, 3, \dots, \quad \alpha_n = \left(\frac{n\pi c}{L}\right)^2$$

and

$$G_0'(t) = f_0$$

Solving these gives the Soln its dynamics.

~~2.2~~ 2.1: $F(x,t) = e^{-t}\sin\left(\frac{2\pi}{L}x\right)$



a Cosine Half-Range
⇒ the dashed curve, and

$$f_0 = \frac{1}{L}\int_0^L F(x,t)\,dt = 0$$

$$f_n = \frac{2}{L}\int_0^L e^{-t}\cos\left(\frac{n\pi}{L}\right)\sin\left(\frac{2\pi}{L}x\right)dx$$

$$\overset{(*)}{=} \begin{cases} \frac{2e^{-t}}{\pi}\left[\frac{(-1)^n+1}{n+2} - \frac{(-1)^{n+1}+1}{n-2}\right], & n \neq 2 \\ 0, & n = 2 \end{cases}$$

[Note: Symmetry + orthogonality will not work as before since the dashed curve isn't exactly a trig fn!]

(*) Use IBP or $\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

Now for (**), [Note: $f_0 = 0 \Rightarrow G_0'(t) = 0 \Rightarrow G_0(t) = C_0 \in \mathbb{R}$

$$G_n' + \alpha_n G_n = f_n = k_n e^{-t}, \quad k_n = \frac{2}{\pi}\left[\frac{(-1)^{n+1}+1}{n+2} - \frac{(-1)^{n+1}+1}{n-2}\right]$$

Note:

$k_{\text{Even}} = 0$

Homogeneous Problem:

$$G_n' = -\alpha_n G_n$$

$$\Rightarrow G_n(t) = B_n e^{-\alpha_n t}$$

Particular Soln:

$$G_n^P(t) = A_n e^{-t}$$

[Again assume $\alpha_n \neq 1$ for all n.

$$\Rightarrow -A_n + \alpha_n A_n = k_n$$

$$\Rightarrow A_n = \frac{k_n}{\alpha_n - 1}$$

$$\Rightarrow G_n(t) = B_n e^{-\alpha_n t} + A_n e^{-t}, \quad \underbrace{n = 1, 2, 3, \cdots}_{\text{well kinda. If } n \text{ is Even then } k_n = 0 \Rightarrow \text{no force! } (*)}$$

(*) Where ~~this~~,

$$A_n = \begin{cases} \dfrac{k_n}{\alpha_n - 1}, & n \equiv \text{odd} \\ 0, & n \equiv \text{Even} \end{cases}$$

Then general sol$^{\underline{n}}$:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} \left[B_n e^{-\alpha_n t} + A_n e^{-t} \right] \cos\left(\frac{n\pi}{L} x\right)$$

If $u(x,0) = g(x)$ then

$$g(x) = C_0 + \sum_{n=1}^{\infty} \left[B_n + A_n \right] \cos\left(\frac{n\pi}{L} x\right)$$

$$\Rightarrow \quad C_0 = \frac{1}{L} \int_0^L g(x)\, dx$$

$$B_n + A_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

$$\Rightarrow \quad B_n = \begin{cases} \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx, & n \equiv \text{even} \\ \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx - \frac{k_n}{\alpha_n - 1}, & n \equiv \text{odd} \end{cases}$$