

For full credit, you must show all work and box answers.

1. Given

$$\begin{aligned}\frac{dx}{dt} &= y(x^2 + y^2 - 1) \\ \frac{dy}{dt} &= -x(x^2 + y^2 - 1)\end{aligned}$$

(a) Is this system linear?

No

(b) Find the equilibrium solutions for this system.

$$\frac{dx}{dt} = y(x^2 + y^2 - 1) = 0$$

$$\begin{aligned}y &= 0 & x^2 + y^2 - 1 &= 0 \\ && x^2 + y^2 &= 1\end{aligned}$$

$$\frac{dy}{dt} = -x(x^2 + y^2 - 1) = 0$$

$$\begin{aligned}y &= 0 & -x(x^2 - 1) &= 0 \\ x &= 0 & x &= \pm 1\end{aligned}$$

EP: $(0,0), (1,0), (-1,0)$
All points on circle $x^2 + y^2 = 1$
 $((1,0), (-1,0))$ are on circle

$x^2 + y^2 = 1$: $-x(0) = 0$
 $0 = 0$
works for all x, y
such that
 $x^2 + y^2 = 1$,
all points on
the circle.

2. Given

$$\begin{aligned}\frac{dx}{dt} &= 2x + y^3 \\ \frac{dy}{dt} &= y\end{aligned}$$

(a) Is this system linear?

No

(b) Find the general solution to the system.

$$\frac{dy}{dt} = y$$

$$y = y_h: \frac{dy_h}{dt} - y_h = 0$$

$$y_h = e^{rt}, y_h' = re^{rt}$$

$$re^{rt} - e^{rt} = 0$$

$$r - 1 = 0$$

$$r = 1, [y_h = y_1 = k_1 e^{rt}]$$

$$\frac{dx}{dt} = 2x + (k_1 e^{rt})^3$$

$$\frac{dx}{dt} - 2x = k_1^3 e^{3rt} + k_1 e^{rt}$$

$$x_h: \frac{dx_h}{dt} - 2x_h = 0$$

$$x_h = e^{rt}, x_h' = re^{rt}$$

$$re^{rt} - 2e^{rt} = 0$$

$$r - 2 = 0, r = 2, x_h = k_2 e^{2t}$$

$$x_p: \frac{dx_p}{dt} - 2x_p = k_2^3 e^{3rt}$$

$$x_p = k_2 e^{3rt}$$

$$x_p' = 3k_2 e^{3rt}$$

$$3k_2 e^{3rt} - 2k_2 e^{3rt} = k_2^3 e^{3rt}$$

$$2 = k_2^3 e^{3rt}$$

$$x_p = k_2^3 e^{2t}$$

$$x(t) = k_1 e^{2t} + k_2^3 e^{3rt}$$

(c) Find the particular solution that satisfies the initial condition $(x(0), y(0)) = (1, 1)$.

$$x(0) = k_1 + k_2^3 = 1, \quad k_1 + 1 = 1, \quad k_1 = 0$$

$$y(0) = k_2 = 1$$

$$[x(t) = e^{3t}, y(t) = ct]$$

* Integrating Factor method also possible

3. Given

$$\begin{aligned}\frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= x\end{aligned}$$

(a) Is this system linear?

$\boxed{\text{Yes}}$

(b) Rewrite the system in matrix-vector form.

$$\boxed{\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \vec{Y}, \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}}$$

or $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

or $\frac{d\vec{Y}}{dt} = A\vec{Y}, A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

(c) Are $\mathbf{Y}_1(t) = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$ and $\mathbf{Y}_2(t) = \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix}$ solutions to this system?

$$\frac{d\vec{Y}_1}{dt} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$\frac{d\vec{Y}_2}{dt} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \vec{Y}_1 = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = \frac{d\vec{Y}_1}{dt}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \vec{Y}_2 = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix} = \begin{pmatrix} 4e^{-t} \\ -e^{-t} \end{pmatrix} \neq \frac{d\vec{Y}_2}{dt}$$

$\boxed{\vec{Y}_1 \text{ is a solution}}$

$\boxed{\vec{Y}_2 \text{ is not a solution.}}$

(d) Are $\mathbf{Y}_1(t) = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$ and $\mathbf{Y}_3(t) = \begin{pmatrix} 12e^{3t} \\ 4e^{3t} \end{pmatrix}$ solutions to this system?

\vec{Y}_1 is a solution, see above.

$$\frac{d\vec{Y}_3}{dt} = \begin{pmatrix} 36e^{3t} \\ 12e^{3t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 12e^{3t} \\ 4e^{3t} \end{pmatrix} = \begin{pmatrix} 36e^{3t} \\ 4e^{3t} \end{pmatrix} = \frac{d\vec{Y}_3}{dt}$$

$\boxed{\vec{Y}_3 \text{ is a solution}}$

(e) Are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_3(t)$ linearly independent?

$$\vec{Y}_1 = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} \neq k \begin{pmatrix} 12e^{3t} \\ 4e^{3t} \end{pmatrix} = k \vec{Y}_3$$

$\boxed{\text{Yes}}$

or $\vec{Y}_1(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\vec{Y}_1(0) + \vec{Y}_2(0) = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$
 $\vec{Y}_2(0) = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$ linearly independent
+ so yes $\vec{Y}_1(t)$

(f) Find the general solution to the system. What principle are you using to do this? (Hint: You do not need to calculate eigenvalues and eigenvectors.)

$$\boxed{\vec{Y}(t) = k_1 \vec{Y}_1(t) + k_2 \vec{Y}_2(t)}$$

$$\boxed{\vec{Y}(t) = k_1 \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} + k_2 \begin{pmatrix} 12e^{3t} \\ 4e^{3t} \end{pmatrix}}$$

Linearity Principle
or Principle
of Superposition

4. Given

$$\begin{aligned}\frac{dx}{dt} &= 5x + 4y \\ \frac{dy}{dt} &= 9x\end{aligned}$$

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} 5 & 4 \\ 9 & 0 \end{pmatrix} \vec{v}$$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 5 & 4 \\ 9 & 0 \end{pmatrix}$$

(a) Is this system linear?

Yes

(b) Find the general solution to the system.

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 9 & -\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-\lambda) - 36 = 0$$

$$\lambda^2 - 5\lambda - 36 = 0$$

$$(\lambda - 9)(\lambda + 4) = 0$$

$$\lambda_1 = 9, \lambda_2 = -4$$

$$\lambda_1 = 9: (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} -4 & 4 \\ 9 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x_1 + 4y_1 = 0$$

$$y_1 = x_1, x_1 = \alpha$$

$$y_1 = \alpha$$

$$\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -4: (A - \lambda_2 I) \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} 9 & 4 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$9x_2 + 4y_2 = 0$$

$$y_2 = -\frac{9}{4}x_2, x_2 = \alpha$$

$$\begin{pmatrix} \alpha \\ -\frac{9}{4}\alpha \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ -9 \end{pmatrix}$$

(c) Find the particular solution that satisfies the initial condition $(x(0), y(0)) = (2, 15)$.

Write your solution as one vector.

$$\vec{v}(0) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 15 \end{pmatrix}$$

$$k_1 + 4k_2 = 2$$

$$k_1 - 9k_2 = 15$$

$$k_1 = -4k_2 + 2$$

$$-4k_2 + 2 - 9k_2 = 15$$

$$k_1 = 4 + 2 = 6$$

$$-13k_2 = 13$$

$$k_2 = -1$$

$$\vec{v}(t) = 6e^{9t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-4t} \begin{pmatrix} 4 \\ -9 \end{pmatrix}$$

$$\vec{v}(t) = \begin{pmatrix} 6e^{9t} \\ 6e^{9t} \end{pmatrix} - \begin{pmatrix} 4e^{-4t} \\ -9e^{-4t} \end{pmatrix}, \boxed{\vec{v}(t) = \begin{pmatrix} 6e^{9t} - 4e^{-4t} \\ 6e^{9t} + 9e^{-4t} \end{pmatrix}}$$