Homework 2 due Wednesday, 26 January 2011
PHGN 570: Physical and Fourier Optics

1. A “mode-locked” laser puts out a train of pulses that are equally spaced, with a pulse spacing that is equal to the round-trip time τ_{rt} = 2 L/c of the pulse in the laser resonator (of length L). For example, if the pulse is Gaussian with a 1/e field half-width of τ_p, we can represent the laser output by
$E(t) \propto comb(t/\tau_{rt}) \otimes \exp\left[-t^2/\tau_p^2\right]$.
    a. Calculate the spectral intensity $\left|\Im\left\{E(t)\right\}_f\right|^2$ for such a beam. What is the relation between τ_{rt} and the spacing of the comb lines in the spectrum? With a good spectrometer, we would have a resolution of about 0.1nm – how short would the cavity have to be for us to be able to resolve the *spacing* of the spectral comb lines? (A typical laser cavity length is about 1.5m.)
    b. Suppose we use an optical shutter to select a burst of 10 pulses. Calculate and describe the change in the output spectrum. You can model the shutter with a rect( ) function, and assume that the edge of the shutter doesn’t coincide with any of the pulses.

    Note: you can do the actual calculations using whatever method you prefer, but you will find it easier to discuss the characteristics of the spectrum in the context of doing the transform of
    $comb(t/\tau_{rt}) \otimes \exp\left[-t^2/\tau_p^2\right]$ and $rect(t/T_s)\left\{comb(t/\tau_{rt}) \otimes \exp\left[-t^2/\tau_p^2\right]\right\}$.
    Illustrate your discussions with plots using different choices of τ_{rt} and τ_p.

2. The central limit theorem states that the repeated self-convolution of a function will tend towards a Gaussian shape.
    a. Numerically calculate the self-convolution of an exponential decay function $f(t) = \exp(-\alpha t)$, to demonstrate the central limit theorem. That is, calculate $f \otimes f \otimes f \otimes \cdots$. Show a few plots to illustrate how the shape evolves (renormalize each plot after each convolution operation).
    b. Note that the central limit theorem does not hold for all functions – give an argument that the auto-convolution of a sinc function will produce another sinc function.

3. Periodic functions:
    a. Derive the Fourier transform of a periodic triangular wave, defined in one period as $f(t) = |t| \quad (-\pi < t < \pi)$. Hint: use the array theorem, and you can make use of the FT of the triangle pulse from HW1.
    b. As a periodic function, you can also calculate a Fourier series for this waveform. How can the two results be related?

4. Given a linear, shift invariant (LSI) system with a rectangle wave input of finite extent described by

$$f(t)=\left[\frac{1}{3}\operatorname{comb}\left(\frac{t}{3}\right)\operatorname{rect}\left(\frac{t}{100}\right)\right]\otimes\operatorname{rect}(t)$$

Using reasonable approximations where appropriate, find the output for each of the following system descriptions. Sketch the transfer function, impulse response, output spectrum, and output in each case.

a. $H(\omega)=\left[\operatorname{rect}\left(\frac{\omega}{30}\right)-\operatorname{rect}(6\omega)\right]e^{-i\omega}$

b. $h(t)=\operatorname{rect}(t-1)$

c. $h(t)=4\operatorname{sinc}^2(2t)-2\operatorname{sinc}^2(t)$

d. $H(\omega)=\operatorname{rect}\left(\frac{\omega}{4}\right)\operatorname{sign}(-\omega)$