Solution of space-dependent rate eqns.

- 4 level system

$$\frac{\partial N}{\partial t} = R_p - WN - \frac{1}{\tau}N \qquad N(\vec{r},t)$$

$$R_p(\vec{r},t)$$

$$\frac{\partial \phi}{\partial t} = \int WN\,dV - \frac{1}{\tau_c}\phi \qquad W = \text{stim. em. rate}$$

$$= \sigma F$$

↳ vol. of gain medium $\qquad F = I/h\nu$

$$I = c\rho/n$$

ρ = local energy density $\propto |E|^2$ $\qquad \rightarrow W = \frac{\sigma c}{n h\nu}\rho$

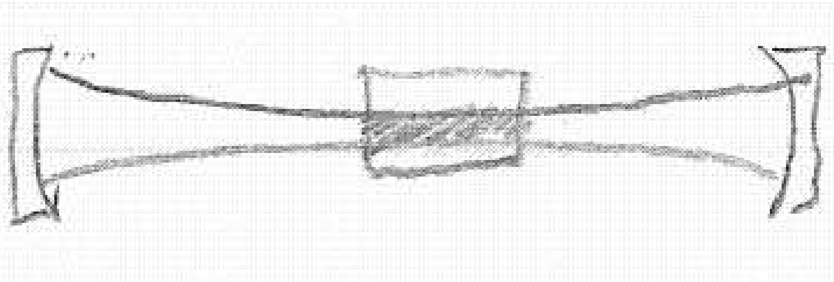
$= \rho_0 |u|^2$ outside $\qquad \rho = n\rho_0|u|^2$ inside

$$\therefore\ \phi = \frac{1}{h\nu}\int \rho\,dV$$

↳ volume of whole cavity.

Two parts

$$= \frac{\rho_0}{h\nu}\Bigg[\underbrace{\int_a n|u|^2 dV}_{\substack{\text{inside}\\ \text{gain}\ \rightarrow\ nV_a}} + \underbrace{\int |u|^2 dV}_{\substack{\text{outside}\\ \text{gain}}}\Bigg] \rightarrow V$$



$u(\vec{r})$ = mode pattern.

normalized so that $\int \rho\,dV$ = total energy in cavity $= \rho_0 V$

$$\therefore\ \rho_0 = (\phi/V)\,h\nu$$

$$\frac{\partial N}{\partial t} = R_p - \frac{\sigma c}{n h\nu}\, n\rho_0|u|^2 N - \frac{1}{\tau}N$$

$$= R_p - \frac{\sigma c}{V}\phi N|u|^2 - \frac{1}{\tau}N$$

$$\frac{\partial \phi}{\partial t} = \int \frac{\sigma c}{n h\nu}\, n\rho_0 |u|^2 N\,dV - \frac{1}{\tau_c}\phi$$

$$= \left[\frac{\sigma c}{V}\int_a N|u|^2 dV - \frac{1}{\tau_c}\right]\phi$$

volume integrals: $|u(r,z)| = \frac{w_0}{w(z)} e^{-r^2/w^2(z)}$

use cavity ABCD for $w(z)$

$$V = \int |u|^2 dV = 2\pi \int_0^\infty \int_0^L |u(r,z)|^2 \, r\,dr\,dz = \int \frac{2\pi w_0^2}{w^2(z)} e^{-\frac{2r^2}{w^2(z)}} r\,dr\,dz$$

let $x = 2r^2/w^2$ $\quad dx = 4r/w^2 \, dr$

$$\rightarrow \int_0^\infty e^{-x} \frac{w^2}{4} dx = \frac{w^2}{4} \quad \therefore$$

z dependence cancels

$V = \frac{\pi w_0^2}{2} \cdot L_e$ $\qquad L_e = L + (n-1)l$

accounting for index weighting.

$V_a = \frac{\pi w_0^2}{2} l$

<u>Threshold:</u> $d\phi/dt = 0 \quad \rightarrow \quad \int_a N |u|^2 dV = \frac{V}{\sigma c \tau_c}$

$\tau_c = \left(\frac{\gamma c}{L}\right)^{-1}$

$$\langle N \rangle \equiv \frac{\int_a N |u|^2 dV}{V_a} \quad \rightarrow \quad \langle N \rangle_c = \underbrace{\frac{V}{V_a}}_{\downarrow \; \frac{L}{l}} \frac{1}{c\sigma\tau_c} = \frac{\gamma}{\sigma l}$$

similarly from $dN/dt = 0$, $\phi = 0$

$\langle R_p \rangle_c = \frac{\langle N \rangle_c}{\tau}$

$$\text{At threshold, } N(\vec{r}) = R_p(\vec{r})\tau = \tau \, \underset{\substack{\downarrow \\ \text{radiative}}}{\eta_r} \underset{\substack{\swarrow \\ \text{transfer}}}{\eta_t} \frac{P_p}{h\nu_p} \frac{2\alpha}{\pi w_p^2} e^{-\frac{2r^2}{w_p^2}} e^{-\alpha z}$$

$$P_{abs} = P_p \eta_r \eta_t \int_0^l \alpha e^{-\alpha z} dz = \eta_r \eta_t (1 - e^{-\alpha l}) P_p$$

$$\rightarrow N(r) = \frac{\tau P_{abs}}{h\nu_p} \frac{2}{\pi w_p^2} e^{-2r^2/w_p^2}$$

critical average inversion dens: assuming no $w(z)$

$$\langle N\rangle_c = \frac{\tau P_{abs}}{h\nu_p}\frac{2}{\pi w_p^2}\frac{\int e^{-2r^2/w_p^2} e^{-2r^2/w_0^2}\, r\,dr}{\underbrace{\int e^{-2r^2/w_0^2}\, r\,dr\,dz}}$$

$$\frac{\left(1/w_p^2+1/w_0^2\right)^{-1}}{w_0^2\, l} = \frac{w_p^2}{w_0^2+w_p^2}\cdot\frac{1}{l}$$

$$\langle N\rangle_c = \frac{\tau P_{abs}}{h\nu_p}\frac{2}{\pi(w_p^2+w_0^2)\,l} = \frac{\gamma}{\sigma l} = \frac{\gamma\tau I_{sat}}{h\nu l} \qquad I_{sat}=\frac{h\nu}{\sigma\tau}$$

threshold absorbed power

$$P_{th} = \frac{\gamma}{2} I_{sat}\,\pi(w_p^2+w_0^2)\frac{h\nu_p}{h\nu}$$

$P_{th}\downarrow$ by reducing w_p and w_0

but w/ one fixed, not much benefit to decreasing other.

Higher-order modes:

$$|u|^2 \sim \left(\frac{w_0}{w(z)}\right)^2 H_l\left(\frac{\sqrt{2}\,x}{w}\right)^2 H_m\left(\frac{\sqrt{2}\,y}{w}\right)^2 e^{-2(x^2+y^2)/w^2}$$

for higher modes $l, m > 0$ less overlap.

$\rightarrow$ higher $P_{th}(l,m)$

$\therefore$ keep $P_{th}(0,0) < P_{abs} < P_{th}(1,0)$ for TEM_{00}

for $w_p = w_0$ $\quad P_{th}(1,0) = 4\,P_{th}(0,0)$

Over-pumping $\rightarrow$ multimode can add hard aperture.

For high power, large mode $\rightarrow$ unstable resonator to force to desired profile.

Or run w/ so many transv. modes $\rightarrow$ smooth.

steady-state ϕ_0:

$$\frac{\partial N}{\partial t} = 0 \quad \rightarrow \quad N\left(1 + \frac{c\sigma\tau}{V}\phi_0 |u|^2\right) = R_p \tau$$

relate ϕ_0 to P_{out}

$$P_{out} = \frac{\gamma_2 c}{2L_e} h\nu \, \phi_0 \quad \rightarrow \quad c\phi_0 = \frac{P_{out}\, 2L_e}{\gamma_2 h\nu}$$

define saturation power

$$P_{sat} = \frac{\gamma_2}{2}\frac{\pi w_0^2}{2}\underbrace{\frac{h\nu}{\sigma\tau}}_{I_{sat}} \quad \rightarrow \quad \sigma\tau = \frac{\gamma_2 \pi w_0^2 h\nu}{4 P_{sat}}$$

$$\rightarrow N\left(1 + \frac{1}{V}\frac{P_{out}\, 2L_e}{\cancel{\gamma_2}\cancel{h\nu}}\frac{\cancel{\gamma_2}\pi w_0^2 \cancel{h\nu}}{4P_{sat}}|u|^2\right) \qquad \frac{V}{L_e} = \frac{\pi w_0^2}{2}$$

$$= N\left(1 + \frac{P_{out}}{P_{sat}}|u|^2\right) = R_p \tau$$

$$\langle N \rangle_0 = \frac{1}{V_a}\int \frac{R_p |u|^2 \tau}{1 + (P_{out}/P_{sat})|u|^2}\, dV = \langle N_c \rangle = \frac{\gamma}{\sigma l}$$

Put in pump, mode distribution $\rightarrow P_{out}$.

η_s slope eff no longer const.