Fractals, Dynamical Systems and Chaos

MATH225 - Field 2008

Outline

- Introduction
- Fractals
- Dynamical Systems and Chaos
- Conclusions

Introduction

- When beauty is abstracted then ugliness is implied. When good is abstracted then evil has been implied. - Lao Tzu
- A mathematical model is an abstraction of a natural/physical system which uses a formal symbolic language to derive knowledge pertaining to the system.
- In abstraction much is lost. However, this does not imply that the knowledge gained from these models is simple.

Geometric Complexity

- What is geometrically straightforward?
  - Euclidian Geometry
  - Non-Euclidian Geometry
- What is not geometrically straightforward?
  - Fractal Geometry

Fractals

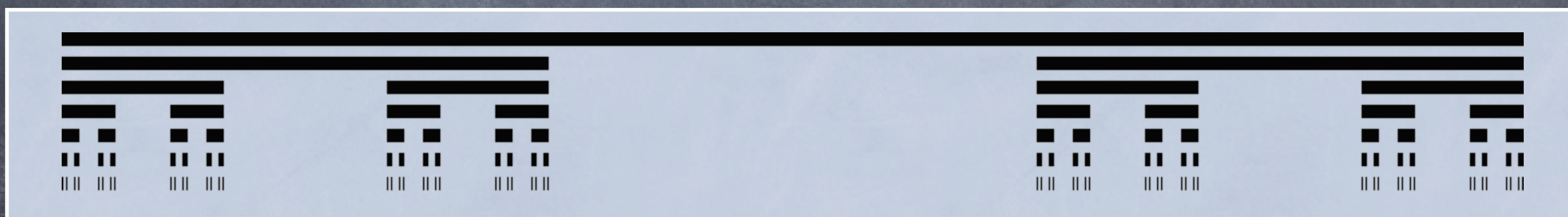
- A fractal is a set with non-integral Hausdorff Dimension, a type of fractal dimension, greater than its Topological Dimension.
  - The topological dimension of the real line is one. The plane is has dimension two.
- Fractals, once thought to be pathological creations, have connections to chaotic dynamical systems.

History of Fractal Geometry

- Cantor Set
- Peano Curves
- Koch Curve
- Gaston Julia
- Benoit Mandelbrot

Cantor Set

- Take the interval [0,1] and remove sub-intervals ad infinitum.

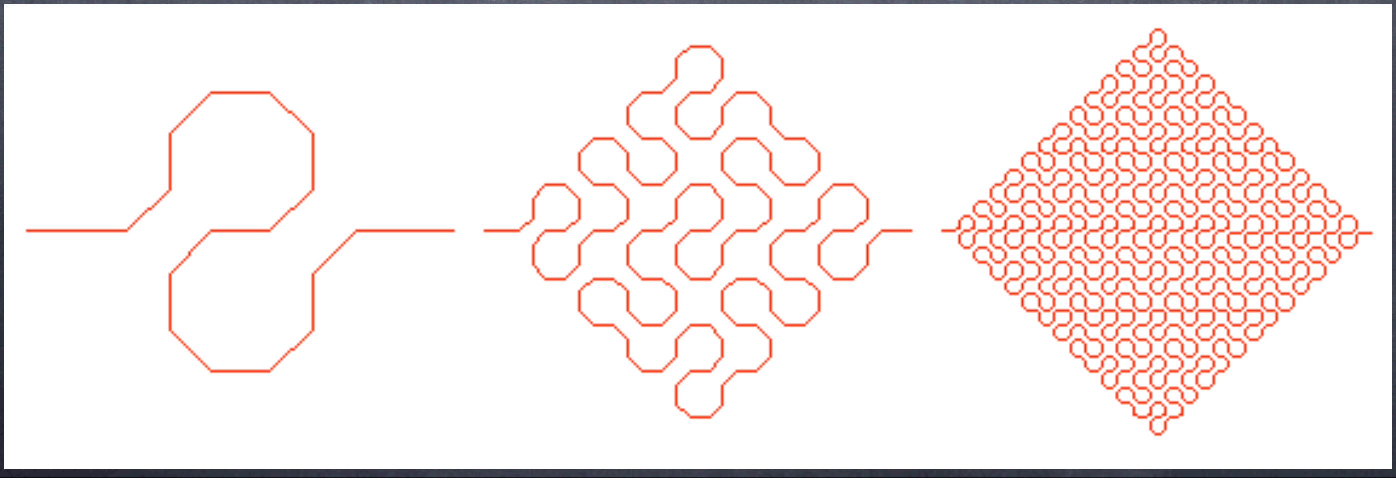


- Introduced in 1883 by Georg Cantor.
- The set of points which is left behind has a topological dimension of 0 and a Hausdorff dimension of approximately 0.6309.

Peano Curve

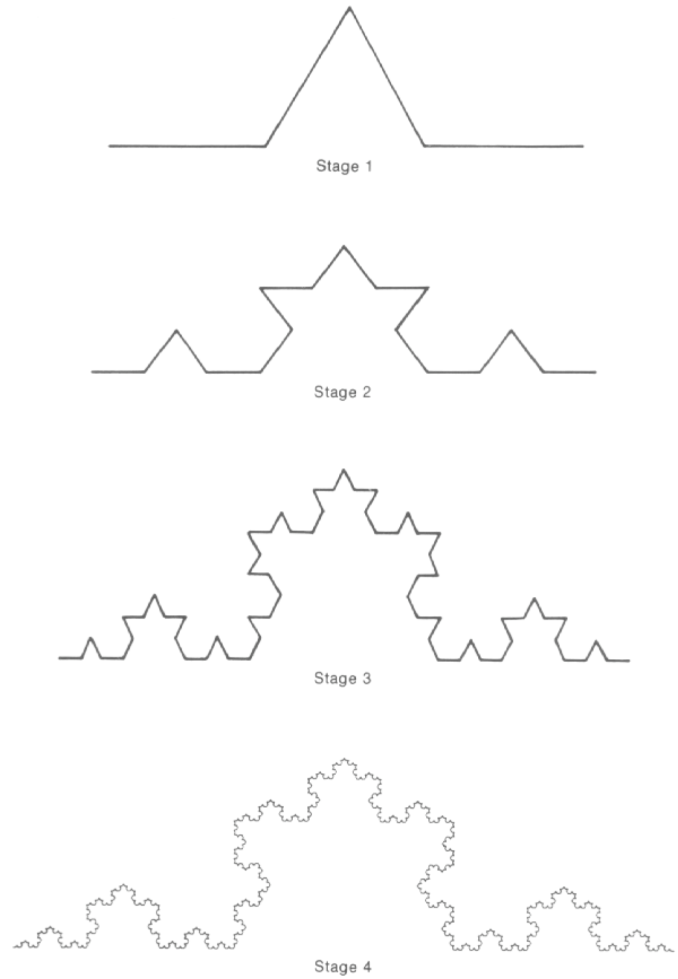
- A curve is the geometric object described by a continuous mapping whose domain is [0,1]. Often the range of such a function is, at most, the entire real line.

- In 1890 Giuseppe Peano constructed a continuous mapping from [0,1] to the unit square.



Koch Curve

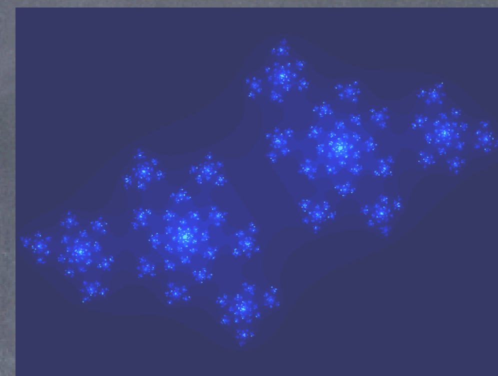
- Helge von Koch in 1904 constructed a curve that is continuous but differentiable nowhere!
- The Koch curve has topological dimension 1 and Hausdorff dimension of approximately 1.26.

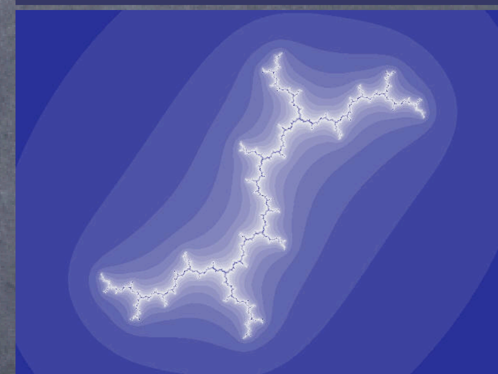


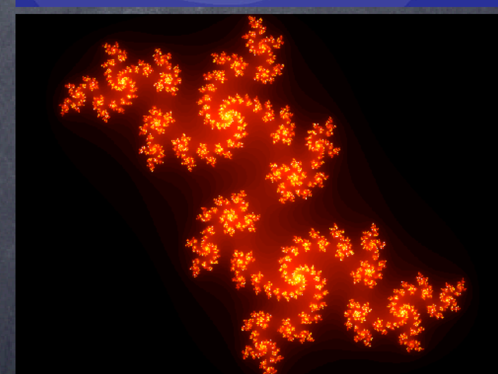
Adapted from Benoit Mandelbrot, *Fractals*.

Gaston Julia's Sets

- Gaston Julia (1893-1978) studies the complex function of complex parameter c, $f_c(z) = z^2 + c$.
- For different values of c the set which is created from the sequence $\{z, f_c(z), f_c(f_c(z)), f_c(f_c(f_c(z))), \ldots\}$ is
  - Disconnected
  - Simply Connected
  - Connected





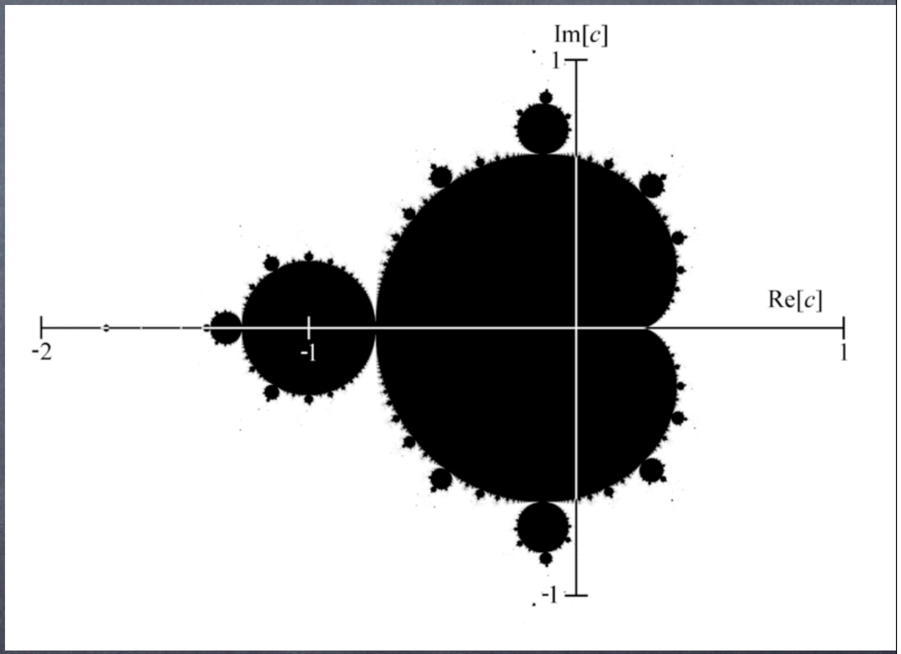


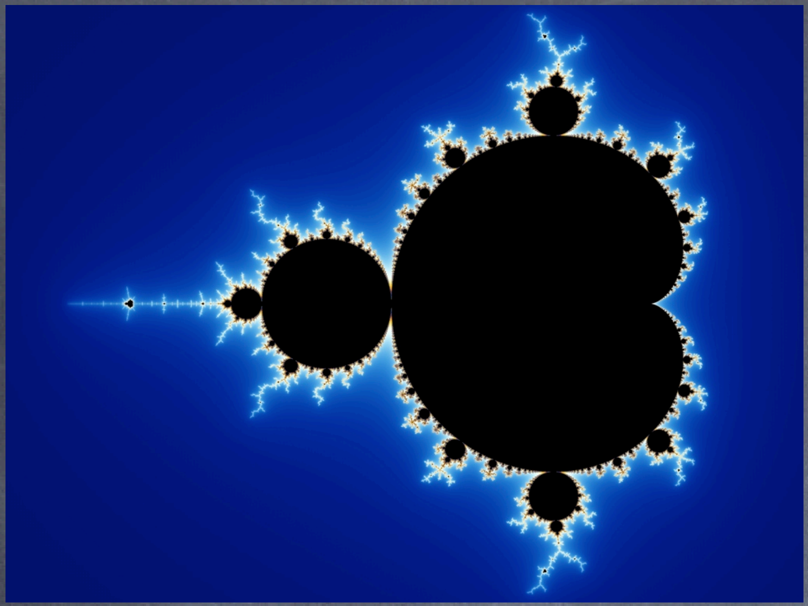
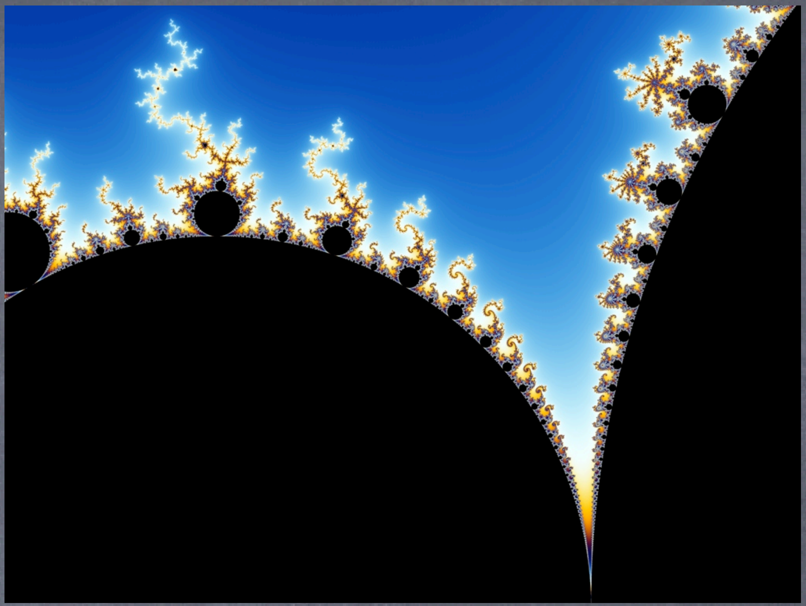
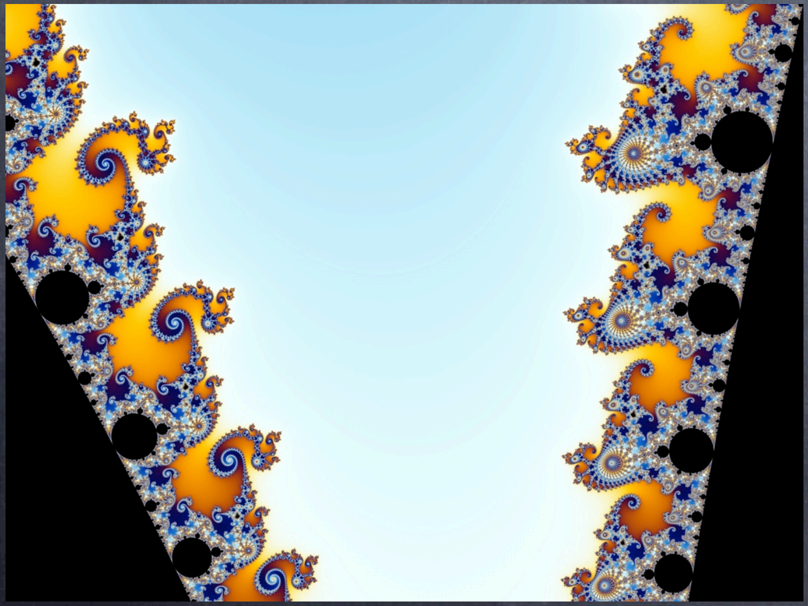
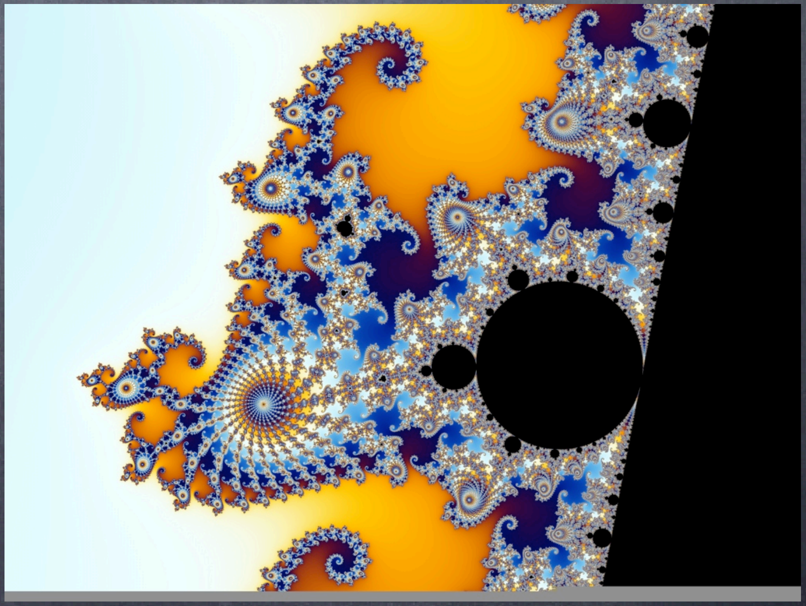
Mandelbrot

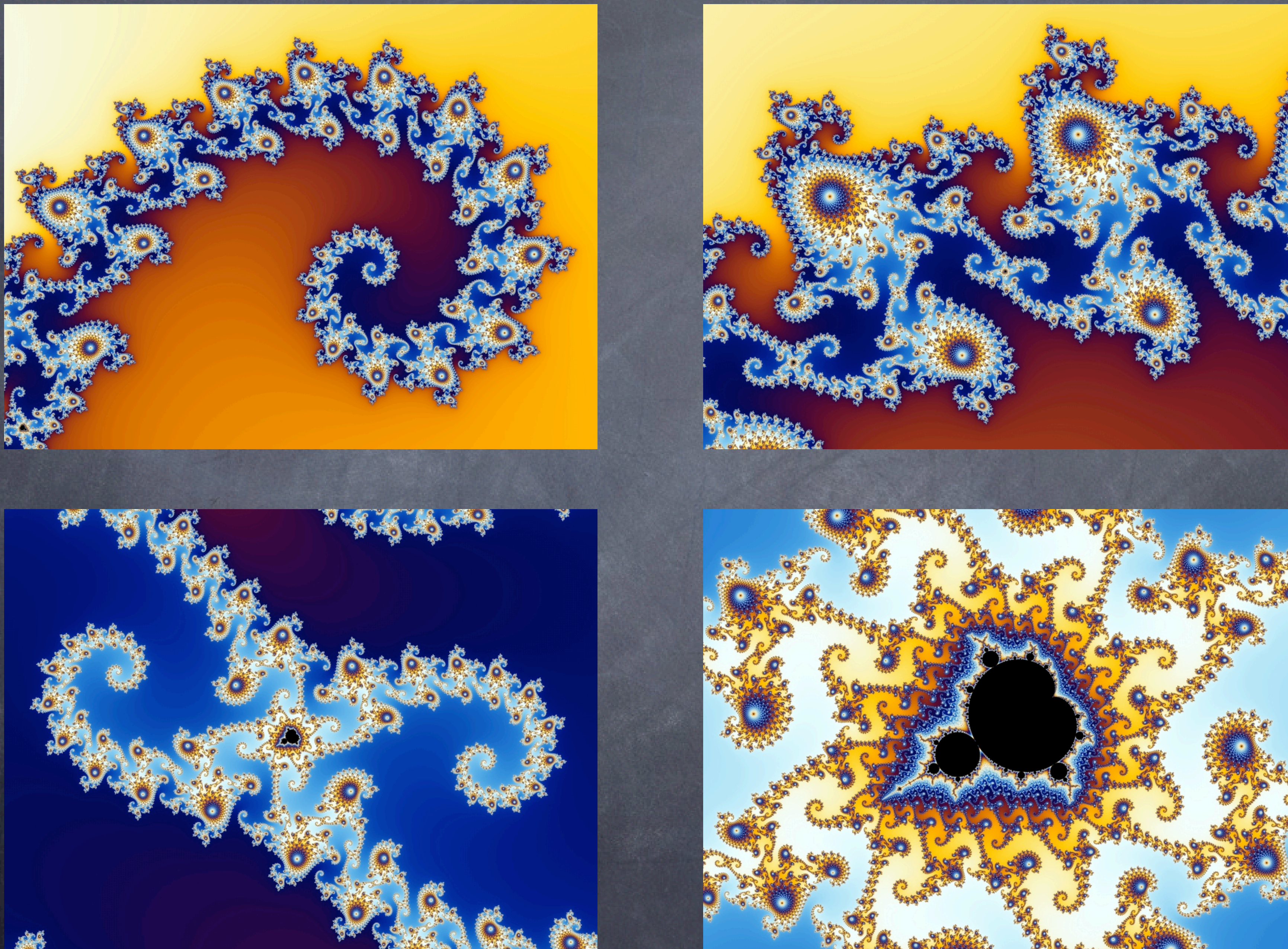
- From 1975-1982 Mandelbrot coins the term 'fractal' an publishes The Fractal Geometry of Nature.
  - Coastline Paradox - Measure the coastline of a land-mass with a ruler of length L. Compare this to the measurement given by a ruler of length l, l<L. - L. F. Richardson (1881-1953)
- Plotting the length of the ruler against the measurement on a log-log scale gives a straight line. The slope of this line is the fractal dimension of the coastline.

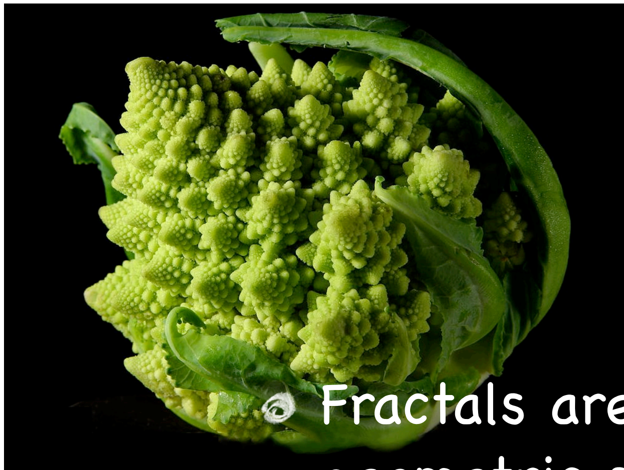
Mandelbrot's Set

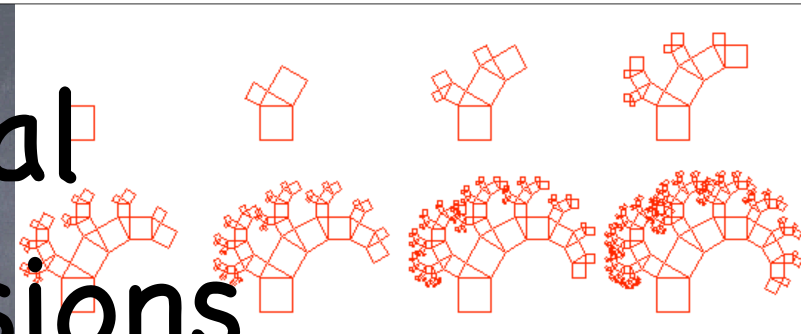
- Mandelbrot set derives from the complex function of complex parameter:
  - $f_c(z) = z^2 + c$
- The behavior of the sequence $\{0, f_c(0), f_c(f_c(0)), f_c(f_c(f_c(0))), \dots\}$ defines the Mandelbrot Set.



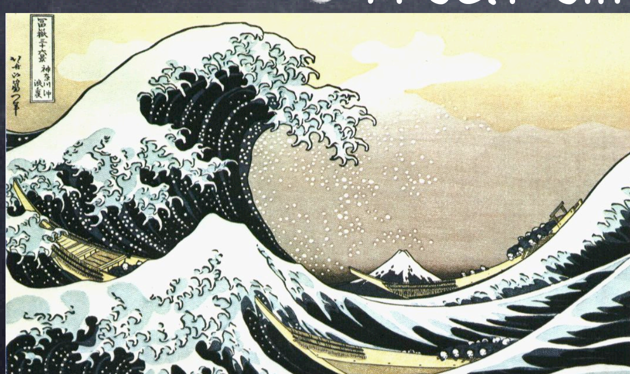


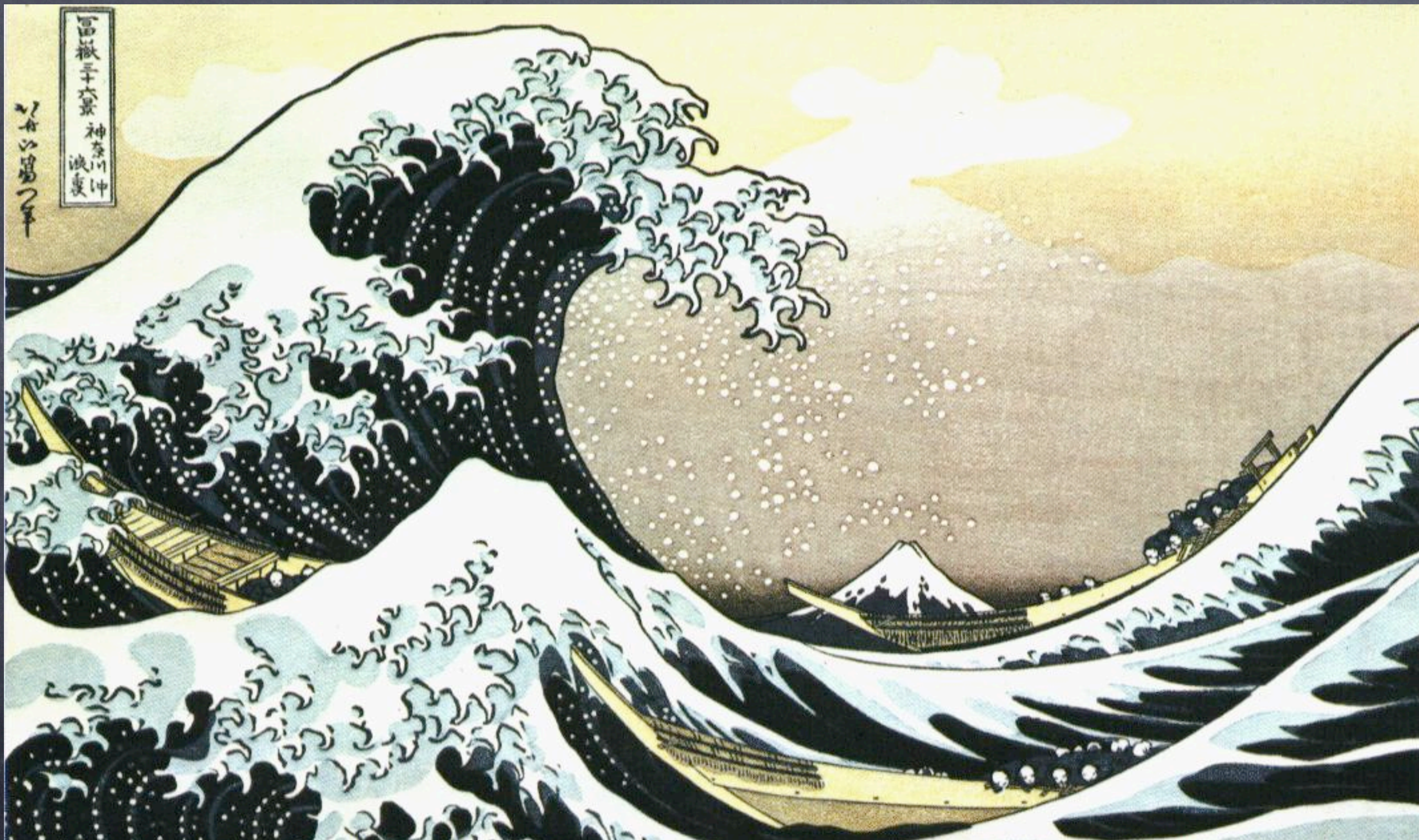




Fractal Conclusions

- Fractals are complicated geometric sets which can be characterized as having:
  - A Hausdorff Dimension greater than its topological dimension.
  - A self-similar structure.

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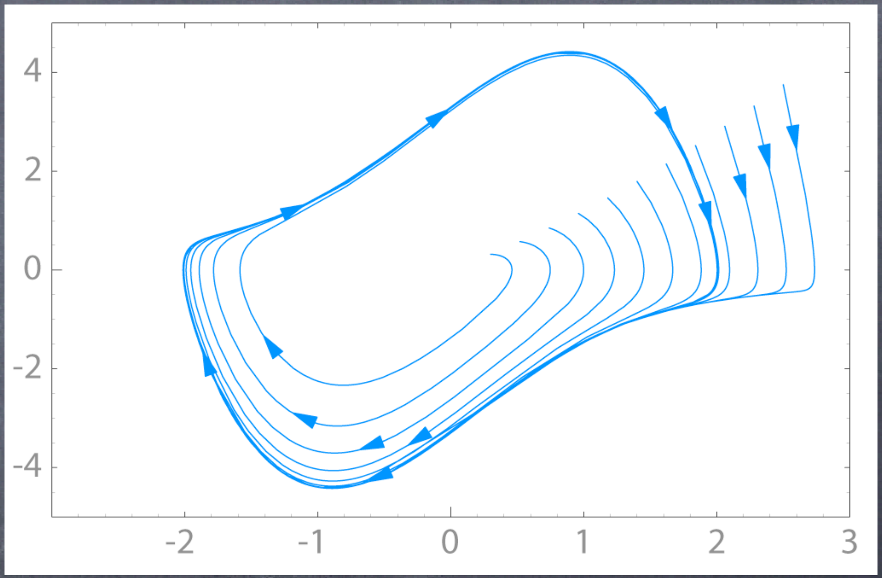
Dynamical Systems

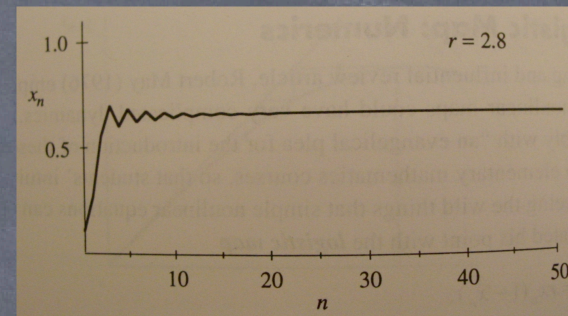
- A dynamical system is the mathematical formalism/rule which describes the deterministic evolution of a point in phase-space.
- The long term behavior of solutions to constant linear dynamical systems is completely understood in terms of eigenvalue/eigenvector decomposition.
- In general, the effects of dissipation evolves trajectories to a steady-state behavior. The geometry of phase space bounding these trajectories is called an attractor.

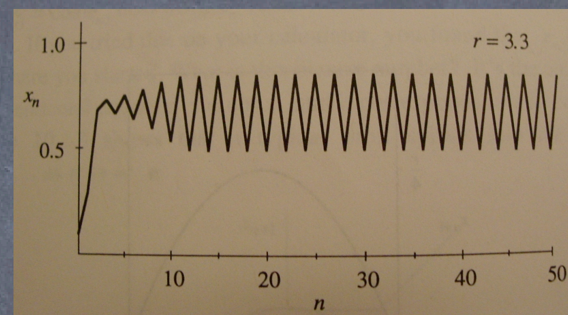
Van der Pol Oscillator

- $y'' - u(1-y^2)y' + y = 0$
  - Electrical circuits using vacuum tubes
  - Action potentials for neurons
  - Seismological modeling of two plates in a geologic fault

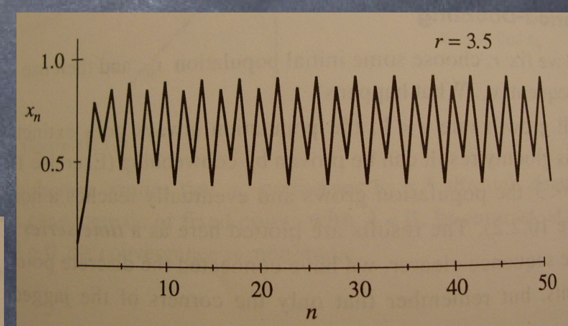


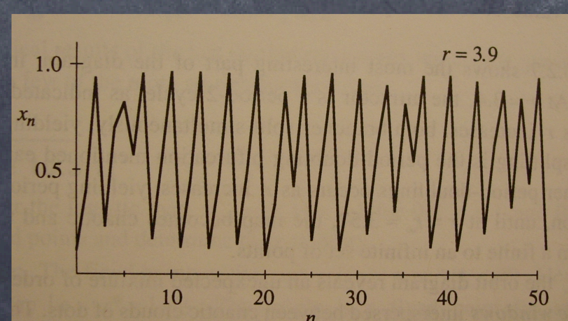
Discrete Logistics Equation





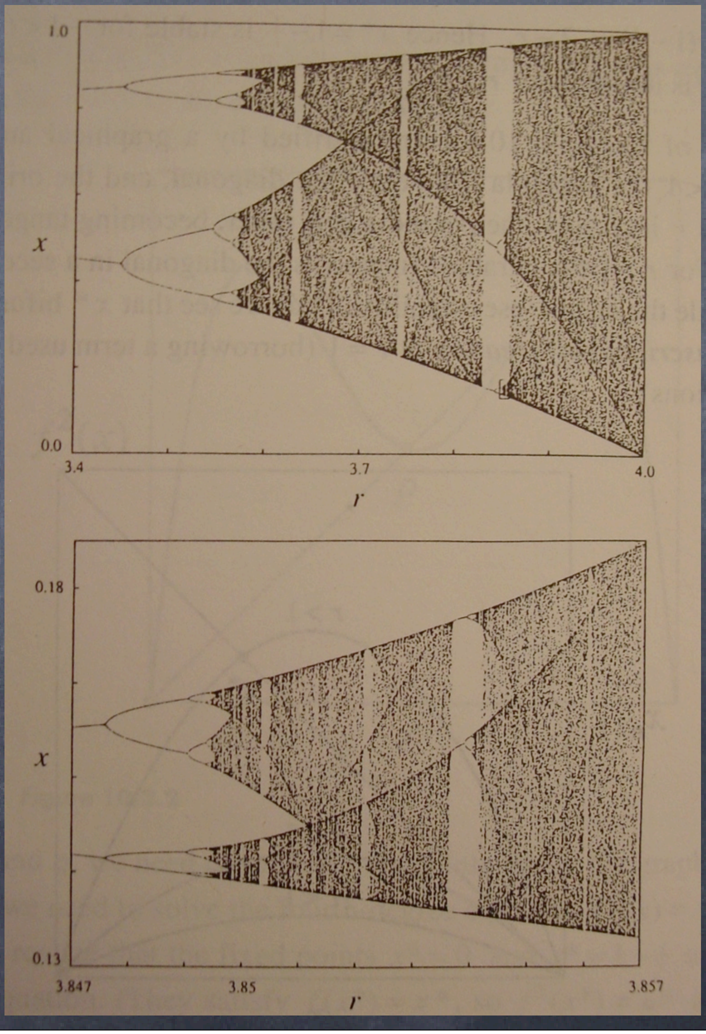
- $x_{n+1} = rx_n(1-x_n)$
  - Models the discrete time evolution of a population subject to a carrying capacity.

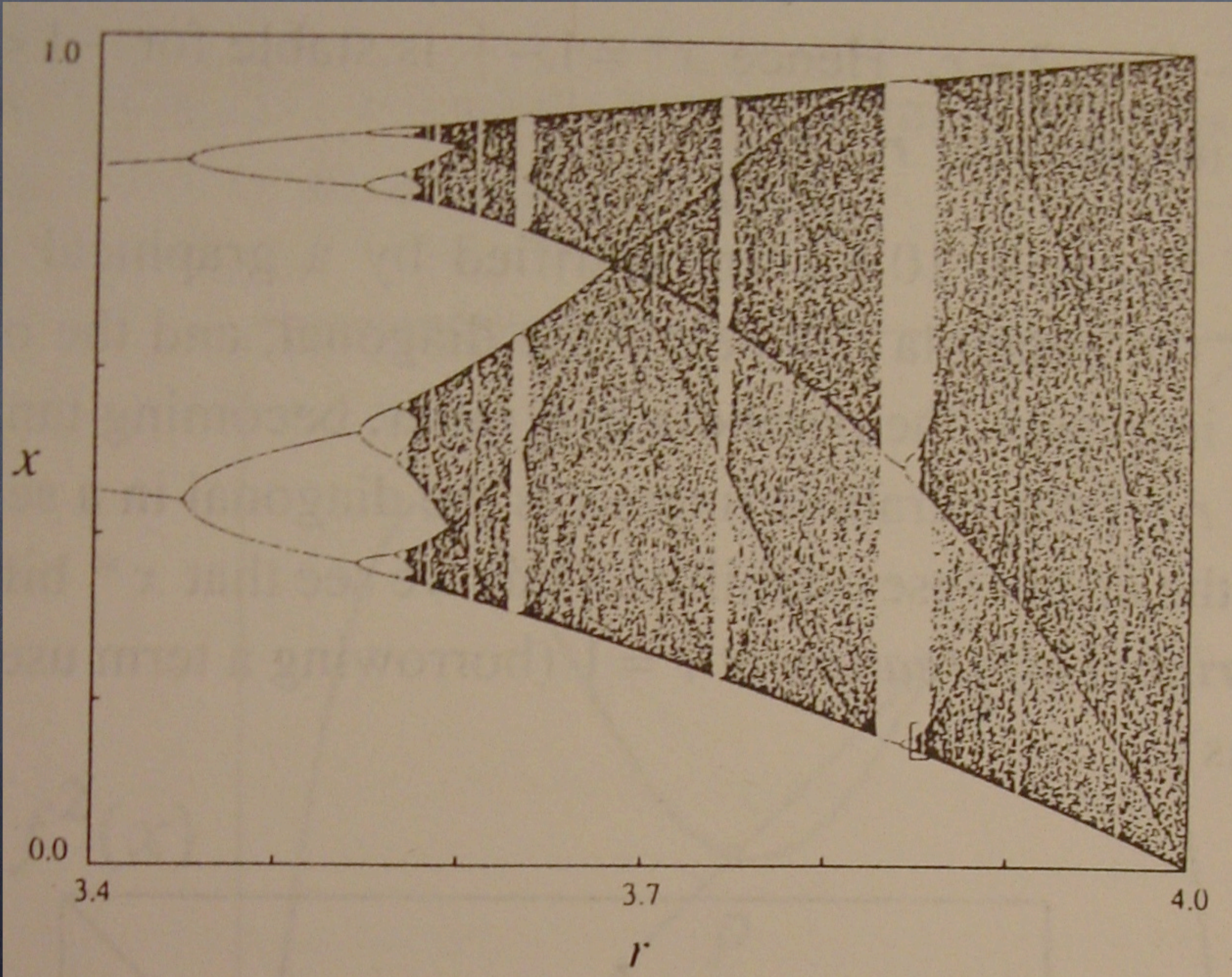


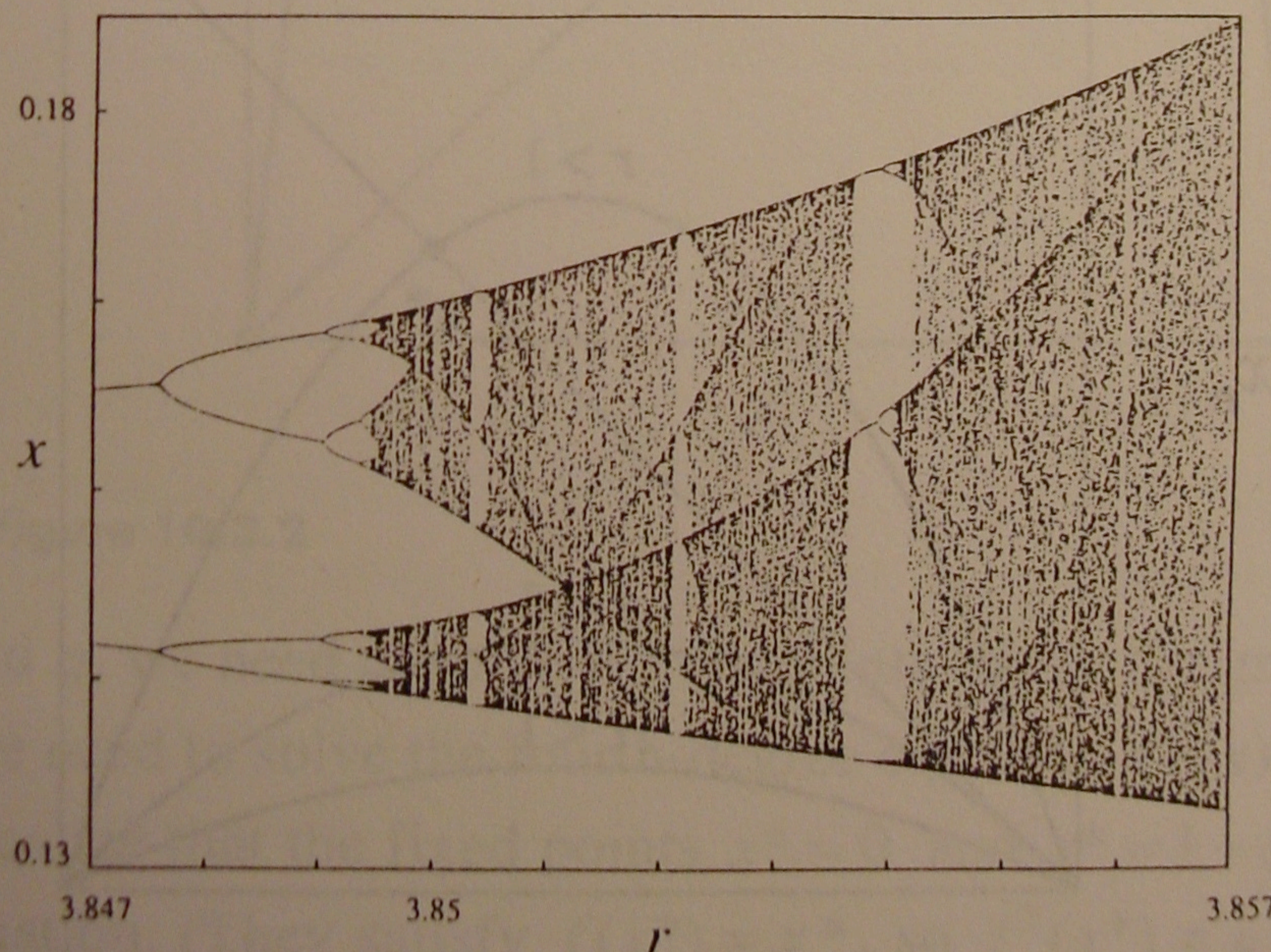


Bifurcation and Period Doubling

- For certain values of the growth rate ‘r’ ‘strange’ things happen to the population evolution.
- Period doubling is a sequence of bifurcations in which the number of equilibrium solutions exhibits unbounded growth.




1.0
0.0
x
3.4
3.7
4.0
r


0.18
x
0.13
3.847
3.85
3.857
r

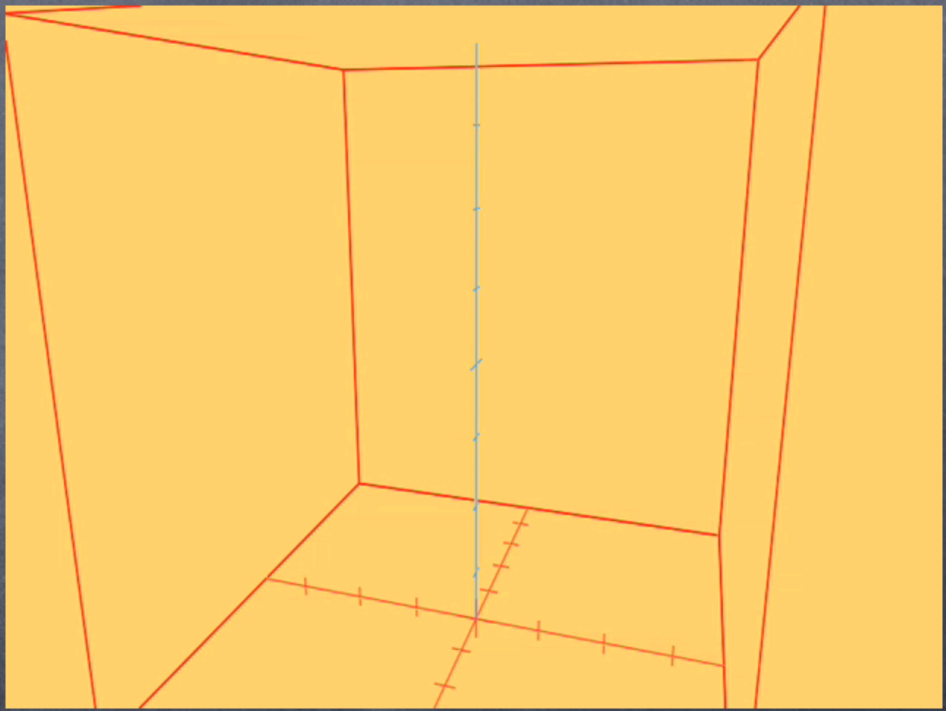
Chaos I

- Period doubling is a typical characteristic of a system entering into chaos.
- For r-values between 3.5 and 4 the logistics map has sensitive dependence on initial conditions, which is another characteristic of chaos.

Lorenz Attractor

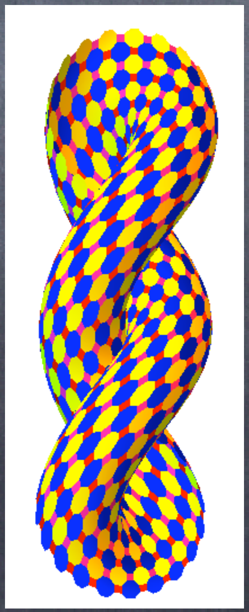
- $x' = a(y-x)$
  $y' = bx-y-xz$
  $z' = -cz+xy$

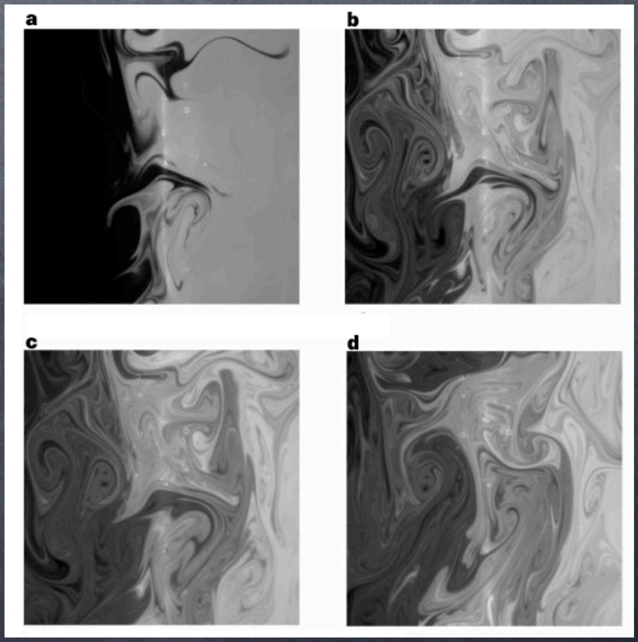
- For $a=10$, $b=28$, $c=8/3$, the Lorenz equations have exponentially divergent trajectories and thus are sensitive to initial conditions.



Chaos II

- The trajectories for the discrete logistics equation and Lorenz equations become so complicated/chaotic because the geometry of the attracting set has been topologically mixed/folded. This generally causes the appearance of fractal geometries.





Chaos III

- If a dynamical system is characterized as having either,
  - a sensitive dependance on initial conditions (chaos),
- or
  - an attracting set with non-integer Hausdorff dimension (fractal),
- then the dynamical systems attracting set is called strange.

Conclusions

- Though mathematical models are approximations of physical/natural systems, math's simplest nonlinear systems give rise to sophisticated and complex behavior,
- Understanding these models and their interpretations requires some of the more advanced geometric, algebraic and numerical techniques currently being studied.
- He accepts the ebb and flow of thing. Nurtures them, but does not own them and lives but does not dwell. - Lao Tzu