Physics 200: Fundamental Equations

Maxwell's Equations

Gauss's Law for Electric Fields: $\oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{Q_{encl}}{\epsilon_0} = 4\pi k Q_{encl}$
Gauss's Law for Magnetic Fields: $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}} = 0$
Electric Flux: $\boldsymbol{\Phi_E} = \int \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$; Magnetic Flux: $\boldsymbol{\Phi_B} = \int \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{A}}$
Ampère/Maxwell: $\oint \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$
Faraday's Law: $\varepsilon_{ind} = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}} = -\frac{d\Phi_B}{dt}$

Fields, Forces and Energy

Electric Field: $d\vec{\boldsymbol{E}} = \frac{kdQ}{r^2}\hat{r} = \frac{kdQ}{r^3}\vec{r}$; $\vec{\boldsymbol{F}}_E = q\vec{\boldsymbol{E}}_{at\ q}$
Electric Potential (Voltage): $V_a - V_b = \int_a^b \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{\ell}}$; $E_x = -\frac{dV}{dx}$; $dV = \frac{kdQ}{r}$
Electrostatic Energy: $U_{of\ q} = qV$
Dielectrics: $\epsilon = \kappa_E \epsilon_0$
Magnetic Field: $d\vec{\boldsymbol{B}} = \frac{\mu_0 I d\vec{\boldsymbol{\ell}} \times \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{\mu_0 I d\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{r}}}{4\pi r^3}$
Magnetic Force: $d\vec{\boldsymbol{F}} = I d\vec{\boldsymbol{\ell}} \times \vec{\boldsymbol{B}}$; $\vec{\boldsymbol{F}}_B = q\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}$
Magnetic Dipole: $\vec{\boldsymbol{\mu}} = NI\vec{\boldsymbol{A}}$; $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\boldsymbol{B}}$

Circuits

Resistors: $dR = \frac{\rho dL}{A}$; $R_{series} = \sum_i R_i$; $R_{parallel} = (\sum_i R_i^{-1})^{-1}$
Capacitors: $C = \frac{Q}{V}$; $U_C = \frac{1}{2}CV^2$; $C_{series} = (\sum_i C_i^{-1})^{-1}$; $C_{parallel} = \sum_i C_i$; $C = \kappa_E C_0$
Ohm's Law: $V = IR$
Current: $I = \frac{dQ}{dt} = n|q|v_d A$
Power: $P = IV$
Kirchoff's Laws: $\sum_{loop} V_i = 0$; $\sum I_{in} = \sum I_{out}$
RC & LR Circuits: Charging and Discharging equations take the form of $e^{-t/\tau}$ and $1 - e^{-t/\tau}$
$\boldsymbol{\tau}_{RC} = RC$ $\boldsymbol{\tau}_{LR} = \frac{L}{R}$
AC Circuits: $X_C = \frac{1}{\omega C}$; $V_C = IX_C$; $Z = \sqrt{R^2 + X_C^2}$; $V = IZ$; $V_{rms} = \frac{V_{peak}}{\sqrt{2}}$
Inductors: $\varepsilon_{ind} = -L\frac{dI}{dt}$; $L = \frac{N\Phi_{\mathrm{B,1\ turn}}}{I}$; $U_L = \frac{1}{2}LI^2$
Inductance: $M_{12} = \frac{N_2 \Phi_{\mathrm{B,1\ turn\ of\ 2}}}{I_{\mathrm{in\ 1}}}$; $\varepsilon_1 = -M_{12}\frac{dI_2}{dt}$

Electromagnetic Waves, Optics and Field Energy Density

Field Energy Density: $u_E = \frac{1}{2}\epsilon_0 E^2$; $u_B = \frac{1}{2\mu_0}B^2$
Momentum: $p = \frac{U}{c}$
Wave Properties: $v = \lambda f$; $k = \frac{2\pi}{\lambda}$; $\omega = 2\pi f$; $B = \frac{E}{c}$
Intensity: $I = c\frac{1}{2}\epsilon_0 E_m^2 = \frac{P_{avg}}{A}$
Reflection/Refraction: $c_1 = \frac{c}{n_1}$; $\theta_{in} = \theta_{out}$; Snell-Descartes Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Additional Information/Useful Constants

Common Electric Fields: $E_{inf\ sheet} = \frac{\sigma}{2\epsilon_0}$; $E_{inf\ line} = \frac{2k\lambda}{r}$; $E_{charged\ ring} = \frac{kQx}{(x^2+a^2)^{3/2}}$; $C_{parallel\ plate} = \frac{\epsilon_0 A}{d}$
Common Magnetic Fields: $B_{inf\ wire} = \frac{\mu_0 I}{2\pi r}$; $B_{solenoid} = \mu_0 n I$; $L_{solenoid} = \mu_0 n^2 A \ell$; $B_{current\ loop} = \frac{\mu_0 N I R^2}{2(x^2+R^2)^{3/2}}$
Fundamental Charge: $e = 1.602 \times 10^{-19}\mathrm{C}$; Electron Mass: $m_e = 9.109 \times 10^{-31}\mathrm{kg}$
Proton Mass: $m_p = 1.673 \times 10^{-27}\mathrm{kg}$
$k = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{\mathrm{Nm^2}}{\mathrm{C^2}}$; $\epsilon_0 = 8.854 \times 10^{-12}\frac{\mathrm{F}}{\mathrm{m}}$
$\mu_0 = 4\pi \times 10^{-7} \approx 12.566 \times 10^{-7}\frac{\mathrm{Tm}}{\mathrm{A}}$
$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$

Vector Formulas

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})\,(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})\,(\mathbf{B}\cdot\mathbf{C})$$

Derivatives of Sums

$$\nabla(f+g) = \nabla f + \nabla g$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$$

Derivatives of Products

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A}$$

$$\nabla\cdot(f\mathbf{A}) = f\,(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(f\mathbf{A}) = f\,(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,(\nabla\cdot\mathbf{B}) - \mathbf{B}\,(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$

Second Derivatives

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla f) = 0$$

Integral Theorems

$$\int_V (\nabla\cdot\mathbf{A})\,dV = \oint_S \mathbf{A}\cdot\hat{\mathbf{n}}\,dS$$ Gauss's (divergence) Theorem

$$\int_S (\nabla\times\mathbf{A})\cdot\hat{\mathbf{n}}\,dS = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell}$$ Stokes's (curl) Theorem

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f)\cdot d\boldsymbol{\ell} = f(\mathbf{b}) - f(\mathbf{a})$$

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right) dV = \oint_S (f\nabla g - g\nabla f)\cdot\hat{\mathbf{n}}\,dS$$ Green's Theorem

Vector Derivatives

Cartesian Coordinates

$$d\boldsymbol{\ell} = \hat{\mathbf{i}}\,dx + \hat{\mathbf{j}}\,dy + \hat{\mathbf{k}}dz\,, \qquad dV = dx\,dy\,dz$$

$$\nabla f = \hat{\mathbf{i}}\,\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{i}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Coordinates

$$d\boldsymbol{\ell} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{k}}dz\,, \qquad dV = r\,dr\,d\phi\,dz$$

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial f}{\partial \phi} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{k}}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates

$$d\boldsymbol{\ell} = \hat{\mathbf{r}}dr + \hat{\boldsymbol{\theta}}r\,d\theta + \hat{\boldsymbol{\phi}}r\sin\theta\,d\phi\,, \qquad dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\,A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\,A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right] + \frac{\hat{\boldsymbol{\theta}}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right] + \frac{\hat{\boldsymbol{\phi}}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Chapter 5, 6, 7, 8 useful relationships

Separation of variables general solutions

Cartesian: $V(x,y,z)$ as combinations of $cos(kx) + sin(kx)$ or $cosh(kx) + sinh(kx)$
or $e^{kx} + e^{-kx}$

Spherical: $V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(cos\,(\theta))$

Cylindrical: $V(r,\varphi) = Aln(r) + B + \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n})(C_n \cos(n\varphi) + D_n \sin(n\varphi))$

***Legendre polynomials*:** $P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{3}{2}x^2 - 1$; $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

Legendre orthogonality relationship: $\int_{-1}^{1} P_n(x)P_l(x)\,dx = \delta_{nl}\frac{2}{2n+1}$

Polarization relationships

$\vec{p} = \alpha\vec{E}$ $\quad \vec{P} = \chi_e\varepsilon_0\vec{E}$ $\quad \varepsilon = \varepsilon_0(1+\chi_e)$ $\quad \kappa = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$

$\sigma_b = \hat{n}\cdot\vec{P}$ $\quad \rho_b = -\vec{\nabla}\cdot\vec{P}$

$\vec{P} = n\vec{p}$ $\quad \alpha = \frac{3\varepsilon_0}{n}\frac{\kappa-1}{\kappa+2}$

$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon\vec{E}$ $\quad \vec{\nabla}\cdot\vec{D} = \rho_f$

Current and current densities

$I = \frac{dQ}{dt} = qn_L v$ $\quad dI = \vec{J}\cdot\overrightarrow{dA}$ $\quad \vec{J} = qn\vec{v}$ $\quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$

$\vec{J} = \sigma\vec{E}$ $\quad dI = \vec{K}\cdot\widehat{e_\perp}dl$ $\quad RC = \frac{\varepsilon}{\sigma}$

Magnetic vector potential

$\vec{B} = \nabla\times\vec{A}$ $\quad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x'})}{|\vec{x}-\vec{x'}|}d^3x$

Chapter 9, 10, 11 useful relationships

Magnetization relationships

$\vec{M} = \chi_m \vec{H} \quad \mu = \mu_0(1 + \chi_m)$

$\vec{J}_b = \nabla \times \vec{M} \quad \vec{K}_b = \vec{M} \times \hat{n}$

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \frac{\vec{B}}{\mu}$

Forms of Faraday's Law

$EMF = -\frac{d\Phi}{dt} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d}{dt}\int \vec{B} \cdot \overrightarrow{dA}$

$\oint \vec{E} \cdot \overrightarrow{dl} = -\int \overrightarrow{\frac{\partial B}{\partial t}} \cdot \overrightarrow{dA}$

Momentum & energy

$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \qquad u_{em} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2 \qquad \frac{d\vec{P}}{dV} = \mu_0 \varepsilon_0 \vec{S}$

Boundary conditions in matter

$D_{2,perp} - D_{1,perp} = \sigma_f \qquad B_{1,perp} = B_{2,perp}$

$\vec{E}_{1,parallel} = \vec{E}_{2,parallel} \qquad \vec{H}_{2,parallel} - \vec{H}_{1,parallel} = \overrightarrow{K_f} \times \hat{n}$

Potentials & Gauges

$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$ *Coulomb:* $\nabla \cdot \vec{A} = 0$

Transforms: $V \to V - \frac{\partial f}{\partial t} \qquad \vec{A} \to \vec{A} + \nabla f$ *Lorentz:* $\nabla \cdot \vec{A} = -\varepsilon_0 \mu_0 \frac{\partial V}{\partial t}$

Chapter 13, 14, 15 useful relationships

Optics

Brewster's angle: $\tan\theta_B = \frac{n_2}{n_1}$ $\quad T = \frac{I_{trans}}{I_{incident}}$ $\quad R = \frac{I_{reflect}}{I_{incident}}$

Phase velocity $v_{ph} = \frac{\omega}{k}$ Group velocity $v_g = \frac{d\omega}{dk}$

Retarded potentials and radiation

Lorentz gauge potentials: $-\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\varepsilon_0}$ $\quad -\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$

Larmor formula: $P = \frac{1}{4\pi\varepsilon_0}\frac{2q^2a^2}{3c^3}$ Scattering cross-section: $\sigma = \frac{P_{avg}}{S_{inc}}$

Lienard-Wiechert: $V(\vec{x},t) = \frac{q}{4\pi\varepsilon_0 R}\frac{1}{1-\hat{n}\cdot\vec{\beta}}$ $\quad \vec{A}(\vec{x},t) = \frac{q\vec{v}}{4\pi\varepsilon_0 c^2 R}\frac{1}{1-\hat{n}\cdot\vec{\beta}}$

Handbook goodies

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ $\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\int \frac{du}{(a+bu^2)^{3/2}} = \frac{u}{a\sqrt{a+bu^2}}$ $\quad \frac{d}{dx}\left(\frac{1}{\text{ta}}\right) = \frac{1}{\sin^2 x}$