Today: Lagrangian Mech: (Ex. from ch. 7)
Wed : Some more (~ → 7.9)

From last time:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = \emptyset$$

↑ generalized

If Q_j could be written as

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j} = Q_j$$

← not necessarily conservative.

$$\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \emptyset \qquad L = T - U$$

If you have some forces that you can't,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_{j,nc}}$$

Other assumption: Forces of constraint did/do no work.

Easiest example! Cartesian coords $\vec{F}_{net} = -\nabla U$

Newtonian

$$F_x = m\ddot{x} = -\frac{\partial U}{\partial x}$$

$$F_y = m\ddot{y} = -\frac{\partial U}{\partial y}$$

Lagrange

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$w = U = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - U$$

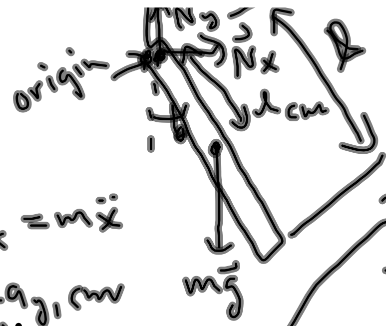
$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} \; ; \; \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -\frac{\partial U}{\partial x} - m\ddot{x} = 0$$

$$m\ddot{x} = -\frac{\partial U}{\partial x}$$

Pendulum:



Newtonian

$$\Sigma F_x = m a_{x,cm} = N_x = m\ddot{x}$$

$$\Sigma F_y = N_y - mg = m a_{y,cm} = m\ddot{y}$$

$$\left\{\begin{aligned} x &= l_{cm}\sin\theta \\ y &= -l_{cm}\cos\theta \end{aligned}\right.$$

$$\dot{x} = l_{cm}\cos\theta\,\dot{\theta}$$

$$\dot{y} = +l_{cm}\sin\theta\,\dot{\theta}$$

$$\ddot{x} = -l_{cm}\sin\theta\,\dot{\theta}^2 + l_{cm}\cos\theta\,\ddot{\theta}$$

$$\ddot{y} = l_{cm}\cos\theta\,\dot{\theta}^2 + l_{cm}\sin\theta\,\ddot{\theta}$$

3 unknowns, 2 eqns.
$\Sigma\tau$ gives 3rd eqn.

Lagrangian

$$T \neq \tfrac{1}{2}mv_{cm}^2$$

$$T = \tfrac{1}{2}I\omega^2$$

$$I = \tfrac{1}{3}ml^2$$

$$T = \tfrac{1}{2}\left(\tfrac{1}{3}ml^2\right)\dot{\theta}^2$$

$$U = mgh = -mgl_{cm}\cos\theta$$

(T does $= \tfrac{1}{2}mv_{cm}^2 + \tfrac{1}{2}I\omega_{cm}^2$)

$$L = T - U = \tfrac{1}{6}ml^2\dot{\theta}^2 + mgl_{cm}\cos\theta$$

$$\frac{\partial L}{\partial\theta} = -mgl_{cm}\sin\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = \tfrac{1}{3}ml^2\dot{\theta}$$

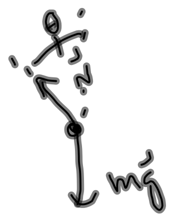
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \tfrac{1}{3}ml^2\ddot{\theta}$$

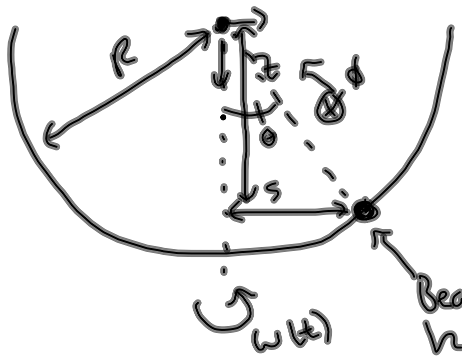
$$\Rightarrow -mgl_{cm}\sin\theta - \tfrac{1}{3}ml^2\ddot{\theta} = \emptyset$$

$$\ddot{\theta} = \frac{-3gl_{cm}}{l^2}\sin\theta$$

kind of a toss up.
($\Sigma\tau$ would work)

Now:





Bead constrained on hoop w/out friction.

$s = R\sin\theta$

$z = R\cos\theta$

Newtonian:

$$\sum F_z = N\cos\theta - mg = m\ddot{z}$$

$$\sum F_s = -N\sin\theta = -m\ddot{s} - ms\dot{\phi}^2 = -m(\ddot{s} + s\omega^2(t))$$

$$\dot{z} = -R\sin\theta\,\dot{\theta}$$

$$\ddot{z} = -R\cos\theta\,\dot{\theta}^2 - R\sin\theta\,\ddot{\theta}$$

$$\dot{s} = R\cos\theta\,\dot{\theta}$$

$$\ddot{s} = -R\sin\theta\,\dot{\theta}^2 + R\cos\theta\,\ddot{\theta}$$

$$N\cos\theta - mg = m(-R\cos\theta\,\dot{\theta}^2 - R\sin\theta\,\ddot{\theta})$$

$$-N\sin\theta = -m(-R\sin\theta\,\dot{\theta}^2 + R\cos\theta\,\ddot{\theta} + R\sin\theta\,\omega^2(t))$$

Lagrangian

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(s\omega)^2 + \tfrac{1}{2}m\dot{s}^2 + \tfrac{1}{2}m\dot{z}^2$$

$$= \tfrac{1}{2}m(R^2\sin^2\theta\,\omega^2(t)) + \tfrac{1}{2}mR^2\cos^2\theta\,\dot{\theta}^2 + \tfrac{1}{2}mR^2\sin^2\theta\,\dot{\theta}^2$$

$$U = -mgR\cos\theta$$

$$L = T - U$$

$$L = \tfrac{1}{2}mR^2\sin^2\theta\,\omega^2 + \tfrac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

$$\frac{\partial L}{\partial \theta} = mR^2\sin\theta\cos\theta\,\omega^2 - mgR\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR^2\ddot{\theta}$$

$$mR^2\sin\theta\cos\theta\,\omega^2 - mgR\sin\theta - mR^2\ddot{\theta} = 0$$

$$\boxed{\ddot{\theta} = -\frac{g}{R}\sin\theta + \omega^2\sin\theta\cos\theta}$$