

## Index ellipsoid

- allows calculation of  $n(\theta, \phi)$

- important for tuning phase matching.

energy density is a scalar:

$$2U_E = \vec{D} \cdot \vec{E} \quad \text{this connects the vector components}$$

$$D_x = \epsilon_0 E_x E_x, D_y = \epsilon_0 E_y E_y, D_z = \epsilon_0 E_z E_z$$

in crystal coord. system.  $E_x = 1/n_x^2$ , etc

$$\therefore 2U_E = \frac{1}{\epsilon_0} \left( \frac{D_x^2}{n_x^2} + \frac{D_y^2}{n_y^2} + \frac{D_z^2}{n_z^2} \right)$$

put into a form like an ellipsoid:

$$\frac{1}{2U_E \epsilon_0} \left( \frac{D_x^2}{n_x^2} + \frac{D_y^2}{n_y^2} + \frac{D_z^2}{n_z^2} \right) = 1$$

define a unit vector  $\hat{d} = \vec{D}/2U_E \epsilon_0 \rightarrow$  points along  $\vec{D}$

$$\frac{d_x^2}{n_x^2} + \frac{d_y^2}{n_y^2} + \frac{d_z^2}{n_z^2} = 1; \quad \text{eqn. of ellipse}$$

For phase matching, we need to know  $\vec{k}$  inside crystal  
 $\rightarrow$  phase, group velocity.

let  $\vec{k} = \vec{k}_0$ . normalized to vacuum  $k_0 = \omega/c$

suppose  $\vec{k}, \vec{D}$  are in x-z plane of crystal  $\vec{k} \perp \vec{D}$

> in a uniaxial crystal,  $n_z = n_0$   $n_x = n_0$

$$\vec{k} = \hat{x} k \sin \theta + \hat{z} k \cos \theta$$

reflective index will be between  $n_x, n_0$

write  $k = n_e(\theta) k_0$ . find  $n_e(\theta)$

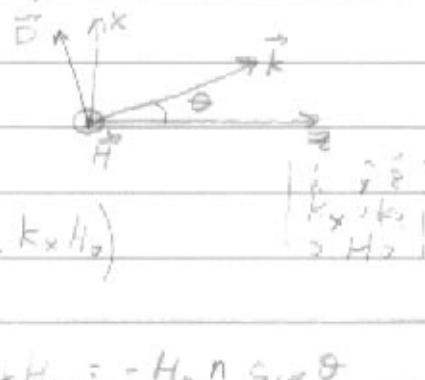
$$\text{get } D_x, D_z \quad \vec{D} = -\frac{1}{\omega} \vec{E} \times \vec{H} = -\frac{1}{\omega} (\hat{x} k_z H_0 + \hat{z} k_x H_0)$$

that corresponds

to  $k_x, k_z$

$$D_x = \frac{k_z H_0}{\omega} = H_0 n \cos \theta \quad D_z = -\frac{k_x H_0}{\omega} = -H_0 \frac{n \sin \theta}{c}$$

the  $n$  here is  $n_e(\theta)$



Put these  $\vec{D}$  components into ellipsoid eqn.

$$\frac{1}{2U_{EE0}} \frac{H_0^2 n^2}{C^2} \left( \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2} \right) = 1$$

$$\text{now magnetic energy is } U_H = \frac{1}{2} \vec{H} \cdot \vec{B} = \frac{1}{2\mu_0} H_0^2$$

so now the multiplier in front is

$$\frac{U_H}{U_E \frac{1}{E_0 \mu_0 C^2}} = \frac{U_H}{U_E} = 1$$

equivalence of  $U_E = U_H$  is shown on next page.

Finally:

$$\frac{1}{n_e^2(\theta)} = \frac{\sin^2 \theta}{n_e^2} + \frac{\cos^2 \theta}{n_o^2}$$

$\theta$  is angle of  $\vec{k}$  to optic axis

Example: uniaxial crystal

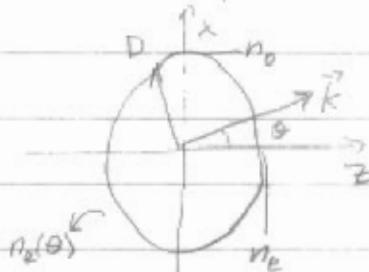
1)  $\vec{k}$  along  $\vec{z}$ : both polarizations  $\rightarrow n = n_o$

2) tilt crystal  $\theta$  away from  $\vec{z}$  in  $x-z$  plane.

$D_y$  still sees  $n_o$

$D_x \rightarrow n_e(\theta)$  given above

where  $\vec{D}$  intersects inner ellipsoid  $\rightarrow n_e(\theta)$



Show that  $U_H = U_E$  in an anisotropic medium  
(Gaussian units)

energy density in magnetic field:

$$8\pi U_H = \vec{H} \cdot \vec{B}$$

put these in terms of  $\vec{E}$  field

$$\vec{k} \times \vec{E} = \frac{\omega}{c} \mu \vec{H} = \frac{\omega}{c} \vec{B}$$

lets write  $\vec{E}/(\omega c) = n \vec{k}$

$$\text{so } \vec{H} \cdot \vec{B} = \frac{n}{\mu} (\vec{k} \times \vec{E}) \cdot n (\vec{E} \times \vec{E})$$

$$\text{vector ID } (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

(see Jackson inside cover)

$$\vec{H} \cdot \vec{B} = \frac{n^2}{\mu} (E^2 - (\vec{k} \cdot \vec{E})^2)$$

$$\text{Now look at } \vec{E} \cdot \vec{D} = 8\pi U_E$$

represent  $\vec{D}$  in terms of  $\vec{E}$

$$n(\vec{k} \times \vec{E}) = \mu \vec{H}$$

$$n \vec{k} \times (\vec{k} \times \vec{E}) = \mu \vec{k} \times \vec{H} = \frac{1}{n} \mu (-\vec{D})$$

$$\therefore n(\vec{k}(\vec{k} \cdot \vec{E}) - \vec{E}) = -\frac{\mu}{n} \vec{D}$$

$$\therefore \vec{E} \cdot \vec{D} = \frac{n^2}{\mu} (E^2 - (\vec{k} \cdot \vec{E})^2)$$

$$\therefore \vec{E} \cdot \vec{D} = \vec{H} \cdot \vec{B} \text{ and } U_E = U_H$$

## Refraction into birefringent materials (Guenther appx BC)

Since  $E_{\parallel}$  is conserved across boundary

$$(\vec{E}_1)_{\parallel} e^{ik_1 \vec{n} \cdot \vec{r} - \omega t} = (\vec{E}_2)_{\parallel} e^{ik_2 \vec{n} \cdot \vec{r} - \omega t}$$

where  $\vec{n}$  is in X-Y plane (on surface)

$$\therefore \text{as usual } \vec{E}_1 \cdot \vec{n} = \vec{E}_2 \cdot \vec{n}$$

$$k_0 n \sin \theta_1 = k_0 n_2 \sin \theta_2$$

for  $n_2 = n_0$  this is normal Snell's law

for  $n_2 = n_e(\theta_2)$ ,

$\rightarrow$  more complicated relationship

$$\text{use } n_e(\theta_2) = \left( \frac{\sin \theta_2}{n_2^2} + \cos \theta_2 \right)^{-1}$$

find  $\theta_2$  that satisfies eqn.

## Birefringent phase matching

It is clear from the solutions of the coupled wave eqns. that high efficiency requires

$$\Delta k = k_1 + k_2 - k_3 = 0$$

When  $\Delta k \neq 0$  the yield varies as

$$\sin^2(\Delta k L/2)$$

$\Delta k = 0$  sets a condition on the refr. index:

$$\Delta k = k_1 + k_2 - k_3 = n_1 w_1 + n_2 w_2 - n_3 w_3$$

as shown in book (2.7.7) this cannot take place with "normal dispersion," where  $n_3 > n_1, n_2$

techniques:

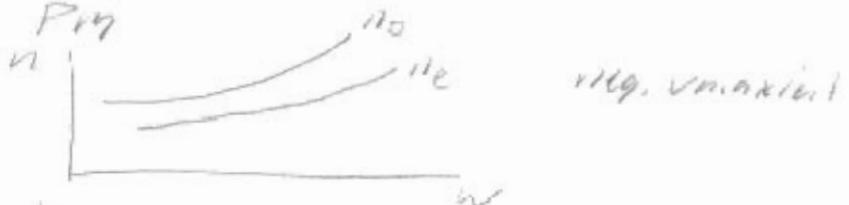
- anomalous dispersion: type was on high side of resonance.



example: mixing in gas (VUV)

H&G

- birefringent POF



→ Angle-tune

→ Temperature tune

- Quasi-phase matching: periodic modulation at  $\Delta k \neq 0$   
→ slow buildup is required

Birefringent case: most common by far.

3 options  $w_2$  is along lowest index direction.

Type I:

$w_1$  and  $w_2$  share same polarization.  
 $\vec{E}_1 \parallel \vec{E}_2$

Type II:

$\vec{E}_1 \perp \vec{E}_2$

"Type III" also called 90° Phase matching

- like Type I, but input at  $\theta = 90^\circ$  and temperature tuned.
- LiNbO<sub>3</sub> most common.

Angle tuning:

use  $n_e(\theta)$  function

$$\frac{1}{n_e(\theta)^2} = \frac{\sin^2 \theta}{n_e^2} + \frac{\cos^2 \theta}{n_i^2}$$

E<sub>k</sub> Type I: negr. variation ( $n_e < n_i$ ) doubling  $\Delta k = 2k$ ,  $-k_2 \Rightarrow 2w_1(n_i, n_e)$   
 $n_e(2w, \theta) = n_o(w)$

input  $\vec{E}_1$  along  $n_o$ , vary crystal angle around

$\vec{E}_2 + \vec{E}_1$  to tune index of  $w_2$



practical concerns:

- cut crystal for the process desired (SHG, mixing...)
  - X-ray diff to get angle close
- protective Al/R coating or in cell to keep off water.

• wavelength separation:

- prism: no background, - polarizer (too expensive)
  - dichroic mirror: typ. refl harmonic at "S"
- $\frac{1}{2}w$                           transm. function, at "P"  
  
 for best efficiency

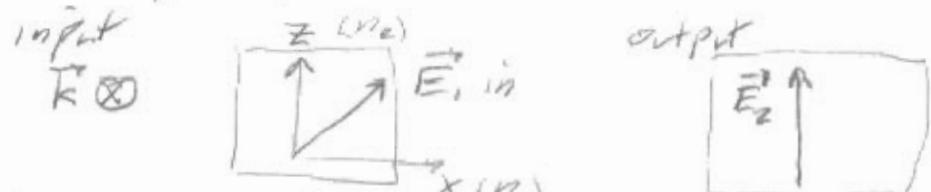
often need multiple mirrors to get rid of fundam.

• temperature stabilization:

- heating to drive off moisture.
- harmonic is often partly absorbed in crystal  
 $\rightarrow \Delta T \rightarrow \Delta k$   
 $\therefore$  stabilize at  $T \approx T_{\text{room}}$ ,
- can tune  $\Delta k$  with  $\Delta T$

Type II: doubling  $\Delta k = \frac{1}{c}(\omega_1 n_e(\theta, \omega_1) + \omega_1 n_o(\omega_1) - \omega_2 n_e(\theta, \omega_2))$

- often seen with KTP higher nonlinearity.
- tripling after type I doubling.
- equal  $E_i$  along  $n_e, n_o$ .



$\therefore$  output at  $45^\circ$  to input.

mirrors out of plane used to flip polarization.  
 (typically)

### Angular acceptance:

input beam must be collimated.

divergent beam has a spread of output directions:



doesn't matter much in one direction

in other  $\rightarrow$  conversion varies across beam

- see line go across beam.

to calculate:

plot  $\text{seinc}^2(\Delta k(\theta_{\text{opt}} + d\theta) L/2)$  vs  $d\theta$

some crystals more forgiving

- Type III is best.

Requisite in several mind. May restrict crystal length

### Pm bandwidth

- short pulses  $\rightarrow$  range  $\Delta w$

- plot  $\text{seinc}^2(\Delta k(w_0 + \Delta w) L/2)$  vs  $\Delta w$

want  $> 90\%$  across input  $Bw$

- restricts length for short pulses.

### Group velocity walk off.

- connected to Pm bandwidth

$$\text{group delay } \tau_g = L/V_g$$

$$\text{if } |\tau_g(w) - \tau_g(2w)| > \tau_p$$

$\rightarrow$  inefficient doubling.