

For full credit, you must show all work and box answers.

1. Find the solutions of the given second-order equations or initial-value problems. Use the method from section 3.6.

(a)  $y'' + 6y' + 9y = 0$

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}$$

$$r^2e^{rt} + 6re^{rt} + 9e^{rt} = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0$$

$$r = -3$$

$$y(t) = k_1 e^{-3t} + k_2 t e^{-3t}$$

(b)  $\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}$$

$$r^2e^{rt} + re^{rt} + e^{rt} = 0$$

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$r = \frac{-1 \pm i\sqrt{3}}{2}$$

$$e^{(-\frac{1}{2} + i\frac{\sqrt{3}}{2})t} = e^{-\frac{t}{2}} e^{i\frac{\sqrt{3}}{2}t} = e^{-\frac{t}{2}} (\cos(\frac{\sqrt{3}}{2}t) + i\sin(\frac{\sqrt{3}}{2}t))$$

$$y(t) = k_1 e^{-\frac{t}{2}} \cos(\frac{\sqrt{3}}{2}t) + k_2 e^{-\frac{t}{2}} \sin(\frac{\sqrt{3}}{2}t)$$

(c)  $4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}$

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}$$

$$4r^2e^{rt} - 8re^{rt} + 3e^{rt} = 0$$

$$4r^2 - 8r + 3 = 0$$

$$y(t) = \frac{5}{2} e^{\frac{t}{2}} - \frac{1}{2} e^{\frac{3t}{2}}$$

$$(2r-1)(2r-3) = 0$$

$$r_1 = \frac{1}{2}, \quad r_2 = \frac{3}{2}$$

$$y(t) = k_1 e^{\frac{t}{2}} + k_2 e^{\frac{3t}{2}}, \quad y'(t) = \frac{k_1}{2} e^{\frac{t}{2}} + \frac{3k_2}{2} e^{\frac{3t}{2}}$$

$$y(0) = k_1 + k_2 = 2, \quad y'(0) = \frac{k_1}{2} + \frac{3k_2}{2} = \frac{1}{2}$$

$$k_1 = 2 - k_2$$

$$k_1 = \frac{5}{2}$$

$$\frac{1}{2} - \frac{5}{2} + \frac{3k_2}{2} = \frac{1}{2}$$

$$k_2 = \frac{-1}{2}$$

2. Consider the harmonic oscillator with the second-order equation  $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$ . ,  $\frac{d^2y}{dt^2} = -\frac{3}{2}\frac{dy}{dt} - \frac{1}{2}y$

- (a) Find the general solution of the second-order equation that models the motion of the oscillator. Use the method from section 3.6.

$$2y'' + 3y' + y = 0$$

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}$$

$$2r^2e^{rt} + 3re^{rt} + e^{rt} = 0$$

$$2r^2 + 3r + 1 = 0$$

$$r = \frac{-3 \pm \sqrt{9-8}}{4}$$

$$r_1 = \frac{-3+1}{4} = -\frac{1}{2}, r_2 = \frac{-3-1}{4} = -1$$

or  $(2r+1)(r+1) = 0$   
 $r_1 = -\frac{1}{2}, r_2 = -1$

$$y(t) = k_1 e^{-\frac{t}{2}} + k_2 e^{-t}$$

- (b) Classify the oscillator.

$$\lambda_1 = -\frac{1}{2} = r_1, \lambda_2 = -1 = r_2$$

Over damped

- (c) Find the particular solution with the initial condition  $y(0) = 0$  and  $v(0) = 3$ .

$$v(t) = y'(t) = -\frac{k_1}{2} e^{-\frac{t}{2}} - k_2 e^{-t}$$

$$y(0) = k_1 + k_2 = 0 \quad v(0) = -\frac{k_1}{2} - k_2 = 3$$

$$k_2 = -k_1 \quad -\frac{k_1}{2} + k_1 = 3$$

$$k_2 = -6 \quad k_1 = 6$$

$y(t) = 6e^{-\frac{t}{2}} - 6e^{-t}$ ,  $v(t) = -3e^{-\frac{t}{2}} + 6e^{-t}$

- (d) What is the long-term behavior (as  $t \rightarrow \infty$ ) of  $y(t)$  and  $v(t)$  from part(c)?

$$\lim_{t \rightarrow \infty} y(t) = 0$$

$$\lim_{t \rightarrow \infty} v(t) = 0$$

position + velocity both decrease to zero.

- (e) Write the first-order system that corresponds to the second-order differential equation given above, find the eigenvalues, and classify the origin.

$$v = \frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -\frac{3}{2}\frac{dy}{dt} - \frac{1}{2}y = -\frac{3}{2}v - \frac{1}{2}y$$

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$$\text{or } \frac{d\vec{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix} \vec{y}, \vec{y} = \begin{pmatrix} y \\ v \end{pmatrix} \text{ or } \frac{d\vec{y}}{dt} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} \vec{y}, \vec{y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

$$\det \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -\frac{3}{2}-\lambda \end{pmatrix} = 0 \rightarrow \lambda^2 + \frac{3}{2}\lambda + \frac{1}{2} = 0$$

$$(\lambda + \frac{1}{2})(\lambda + 1) = 0 \quad \lambda_1 = -\frac{1}{2}, \lambda_2 = -1$$

\* Same as r

Sink

3. Find the solutions of the given second-order equations or initial-value problems.

$$(a) y'' - 3y' - 4y = 4t^2 - 1$$

$$Y_h: Y_h'' - 3Y_h' - 4Y_h = 0$$

$$Y_h = e^{rt}$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$$r_1 = 4, r_2 = -1$$

$$Y_h = k_1 e^{4t} + k_2 e^{-t}$$

$$Y(t) = k_1 e^{4t} + k_2 e^{-t} - t^2 + \frac{3}{2}t - \frac{11}{8}$$

$$(b) \frac{d^2y}{dt^2} - 4y = t^2 + 3e^t, \quad y(0) = 0, \quad \left. \frac{dy}{dt} \right|_{t=0} = 2$$

$$Y_h: Y_h'' - 4Y_h = 0$$

$$Y_h = e^{rt}$$

$$r^2 - 4 = 0$$

$$r^2 = 4$$

$$r = \pm 2$$

$$Y_h = k_1 e^{-2t} + k_2 e^{2t}$$

$$Y(t) = k_1 e^{-2t} + k_2 e^{2t} - \frac{1}{4}t^2 - \frac{1}{8}e^t$$

$$y'(t) = -2k_1 e^{-2t} + 2k_2 e^{2t} - \frac{1}{2}t - e^t$$

$$y(0) = k_1 + k_2 - \frac{1}{8} - 1 = 0 \quad y'(0) = -2k_1 + 2k_2 - \frac{1}{2} = 2$$

$$k_1 = \frac{9}{8} - k_2$$

$$-\frac{9}{8} + 2k_2 + 2k_2 = 3$$

$$k_1 = \frac{-3}{16}$$

$$(c) y'' - 4y' + 4y = 3e^{2t}$$

$$Y_h: Y_h'' - 4Y_h' + 4Y_h = 0$$

$$Y_h = e^{rt}$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2$$

$$Y_h = k_1 e^{2t} + k_2 t e^{2t}$$

$$\alpha = \frac{3}{2}, \quad Y_p = \frac{3}{2} t^2 e^{2t}$$

$$Y(t) = k_1 e^{2t} + k_2 t e^{2t} + \frac{3}{2} t^2 e^{2t}$$

$$Y_p: Y_p'' - 3Y_p' - 4Y_p = 4t^2 - 1$$

$$Y_p = at^2 + bt + c$$

$$Y_p' = 2at + b, \quad Y_p'' = 2a$$

$$2a - 3(2at + b) - 4(at^2 + bt + c) = 4t^2 - 1$$

$$-4at^2 + (-6a - 4b)t + (2a - 3b - 4c) = 4t^2 - 1$$

$$-4a = 4$$

$$a = -1$$

$$-6a - 4b = 0$$

$$6 - 4b = 0$$

$$b = \frac{3}{2}$$

$$2a - 3b - 4c = -1$$

$$-2 - \frac{9}{2} - 4c = -1$$

$$-4c = \frac{11}{2}$$

$$c = -\frac{11}{8}$$

$$Y_p = -t^2 + \frac{3}{2}t - \frac{11}{8}$$

$$Y_p: Y_p'' - 4Y_p = t^2 + 3e^t$$

$$Y_p = at^2 + bt + c + \alpha e^t$$

$$Y_p' = 2at + b + \alpha e^t$$

$$Y_p'' = 2a + \alpha e^t$$

$$2a + \alpha e^t - 4(at^2 + bt + c + \alpha e^t) = t^2 + 3e^t$$

$$-4at^2 - 4bt + (2a - 4c) + (\alpha - 4\alpha)e^t = t^2 + 3e^t$$

$$-4a = 1$$

$$a = -\frac{1}{4}$$

$$-4b = 0$$

$$b = 0$$

$$2a - 4c = 0$$

$$-\frac{1}{2} - 4c = 0$$

$$-3\alpha = 3$$

$$\alpha = -1$$

$$c = -\frac{1}{8}$$

$$Y_p = -\frac{1}{4}t^2 - \frac{1}{8} - e^t$$

$$Y(t) = -\frac{3}{16}e^{-2t} + \frac{21}{16}e^{2t} - \frac{1}{4}t^2 - \frac{1}{8} - e^t$$

$$Y_p: Y_p'' - 4Y_p' + 4Y_p = 3e^{2t} \quad \text{same as double root}$$

$$Y_p = \alpha e^{2t} Y_p' = \alpha t e^{2t}$$

$$Y_p' = 2\alpha t e^{2t} Y_p'' = 2\alpha t^2 e^{2t}$$

$$Y_p'' = 4\alpha t^2 e^{2t} Y_p''' = 2\alpha t^3 e^{2t} + 2\alpha t e^{2t}$$

$$4\alpha t^2 e^{2t} - 8\alpha t^2 e^{2t} + 4\alpha t^2 e^{2t} = 3\alpha t^2 e^{2t}$$

$$4\alpha t^2 e^{2t} + 4\alpha t^2 e^{2t} - 4\alpha t^2 e^{2t} = 0 \neq 3\alpha t^2 e^{2t}$$

$$4\alpha t^2 e^{2t} + 4\alpha t^2 e^{2t} - 4\alpha t^2 e^{2t} = 0 \neq 3\alpha t^2 e^{2t}$$

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$$4\alpha t^2 e^{2t} + 4\alpha t^2 e^{2t} - 4\alpha t^2 e^{2t} = 0 \neq 3\alpha t^2 e^{2t}$$

4. Find the solutions of the given second-order equations or initial-value problems.

(a)  $y'' + y = 3 \sin(2t)$ ,  $y(0) = y'(0) = 0$

$$Y_h: Y_h'' + Y_h = 0$$

$$Y_h = e^{rt}$$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$e^{rt} = e^{it} = \cos t + i \sin t$$

$$Y_h = k_1 \cos t + k_2 \sin t$$

$$y(t) = k_1 \cos t + k_2 \sin t - \sin(2t)$$

$$y'(t) = -k_1 \sin t + k_2 \cos t - 2 \cos(2t)$$

$$y(0) = k_1 = 0, \quad y'(0) = k_2 - 2 = 0$$

$$k_2 = 2$$

$$\boxed{y(t) = 2 \sin t - \sin(2t)}$$

(b)  $y'' + 6y' + 5y = 4e^{-t} \cos(3t)$ ,

$$Y_h: Y_h'' + 6Y_h' + 5Y_h = 0$$

$$Y_h = e^{rt}$$

$$r^2 + 6r + 5 = 0$$

$$(r+5)(r+1) = 0$$

$$r_1 = -5, \quad r_2 = -1$$

$$Y_h = k_1 e^{-5t} + k_2 e^{-t}$$

$$y(t) = k_1 e^{-5t} + k_2 e^{-t}$$

$$-\frac{36}{225} e^{-t} \cos(3t) + \frac{48}{225} e^{-t} \sin(3t)$$

$$Y_c: Y_c'' + Y_c = 3e^{2it} = 3 \cos(2t) + i \underbrace{3 \sin(2t)}_{\text{Imag.}}$$

$$Y_c = \alpha e^{2it}$$

$$Y_c' = 2i\alpha e^{2it}, \quad Y_c'' = -4\alpha e^{2it}$$

$$-4\alpha e^{2it} + \alpha e^{2it} = 3e^{2it}$$

$$-3\alpha e^{2it} = 3e^{2it}$$

$$\alpha = 1$$

$$Y_c = e^{2it} = -\cos(2t) + i \underbrace{\sin(2t)}_{\text{Imag.}}$$

$$Y_p = -\sin(2t)$$

$$Y_c: Y_c'' + 6Y_c' + 5Y_c = 4e^{-t} e^{3it} = 4e^{(-1+3i)t}$$

$$= 4e^{-t} (\underbrace{\cos(3t)}_{\text{Real}} + i \sin(3t))$$

$$Y_c = \alpha e^{(-1+3i)t}$$

$$Y_c' = (-1+3i)\alpha e^{(-1+3i)t}$$

$$Y_c'' = (-1+3i)^2 \alpha e^{(-1+3i)t}$$

$$= (-8-6i)\alpha e^{(-1+3i)t}$$

$$(-8-6i)\alpha e^{(-1+3i)t} + 6(-1+3i)\alpha e^{(-1+3i)t}$$

$$+ 5\alpha e^{(-1+3i)t} = 4e^{(-1+3i)t}$$

$$(-9+12i)\alpha e^{(-1+3i)t} = 4e^{(-1+3i)t}$$

$$\alpha = \frac{4}{-9+12i} \left( \frac{-9-12i}{-9-12i} \right) = \frac{-36-48i}{81+144}$$

$$\alpha = \frac{-36}{225} - \frac{48}{225} i$$

$$Y_c = \left( \frac{-36}{225} - \frac{48}{225} i \right) e^{-t} (\cos(3t) + i \sin(3t))$$

$$Y_c = \underbrace{\frac{-36}{225} e^{-t} \cos(3t) + \frac{48}{225} e^{-t} \sin(3t)}_{\text{Real}} + i \underbrace{e^{-t} \left( \frac{-48}{225} \cos(3t) - \frac{36}{225} \sin(3t) \right)}_{\text{Imag.}}$$