**MATH 225 - Differential Equations** June 9 , 2008
**Homework 8, Field 2008** **Due Date:** June 12, 2008

LAPLACE TRANSFORMS - SECOND-ORDER LINEAR EQUATIONS (REVISITED) - FORCED MASS-SPRING SYSTEMS

1. Calculate the Laplace transform of $f_1(t) = \sinh(at)$ and $f_2(t) = \cosh(at), a \in \mathbb{R}$.
   **Hint:** You may want to refer to the previous homework assignment for the definitions of $\sinh(x)$ and $\cosh(x)$ or you might find it more efficient to note that $-i\sin(ix) = \sinh(x)$ and $\cos(ix) = \cosh(x)$ and repeat the calculations we did in class to find $\mathfrak{L}\{\cos(kt)\}$ and $\mathfrak{L}\{\sin(kt)\}$, which will find the transform of $f_1$ and $f_2$ simultaneously. Doing it this way should make the standard form for these transforms, `http://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms`, make more sense.

2. Using the definition of transform show the following relationships:
   (a) $\mathfrak{L}\{e^{at}f(t)\} = F(s-a)$
   (b) $\mathfrak{L}\{f(t-a)u_a(t)\} = e^{-as}F(s), \quad a \geq 0$
   (c) $\mathfrak{L}\{f(t)u_a(t)\} = e^{-as}\mathfrak{L}\{f(t+a)\}, \quad a \geq 0.$

3. Consider the following second-order linear ordinary differential equation with constant coefficients,
$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \delta(t), \quad y(0) = 0, \; y'(0) = 0. \tag{1}$$
   Solve the IVP (1) for the following cases:
   (a) $a = 1, b = -2, c = -3$
   (b) $a = 1, b = 4, c = 4$
   (c) $a = 1, b = -4, c = 13$
   (d) $a = 1, b = 0, c = 9$

4. Given the following forced simple harmonic oscillator.
$$2\frac{d^2y}{dt^2} + 8y = 6\cos(\omega t), \quad y(0) = 1, \; y'(0) = -1. \tag{2}$$
   (a) Set $\omega = 1$ and find the solution to the initial value problem.
   (b) Set $\omega = 2$ and find the solution to the initial value problem.
   (c) Describe the differences in the long term behavior of the steady-state solution for each oscillator

5. Again we investigate the forced mass spring system given by,
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}. \tag{3}$$
   (a) Suppose we have that $b = 0$, $y(0) = \alpha$, $y'(0) = \beta$ and $f(t) = A\delta_T(t)$, show that the solution to (3) subject to these constraints is given by,
$$y(t) = \alpha\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t) + \frac{A}{m\omega}u_T(t)\sin(\omega(t-T)), \tag{4}$$
   where $\omega^2 = \dfrac{k}{m}$.
   (b) Suppose that we wish to hit the mass in such a way that after the impact the oscillations stop. Show that for this to occur we must choose,
$$A = \frac{\alpha m\omega}{\sin(\omega T)} \tag{5}$$
$$T = \frac{1}{\omega}\arctan\left(-\frac{\alpha\omega}{\beta}\right). \tag{6}$$

1) $\mathcal{L}\{e^{ibt}\} = \int_0^\infty e^{ibt} e^{-st}\, dt = \frac{s}{s^2+b^2} + i \frac{b}{s^2+b^2}$

$\Rightarrow \quad \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$

$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$

$\Rightarrow \qquad b = ia$

$\mathcal{L}\{\cos(bt)\} = \mathcal{L}\{\cos(iat)\} = \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2+b^2} =$

$= \frac{s}{s^2+(ia)^2} = \frac{s}{s^2-a^2}$

and $\qquad b = ia$

$-i\mathcal{L}\{\sin(bt)\} = -i\,\mathcal{L}\{\sin(iat)\} = \mathcal{L}\{\sinh(at)\} = \frac{-ib}{s^2+b^2} =$

$= -\frac{i\,(ia)}{s^2+(ia)^2} = \frac{a}{s^2-a^2}$

2)

$-u = a - s$

a) $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t)\, e^{-st}\, dt = \int_0^\infty f(t) e^{-ut}\, dt =$

$= F(u) = F(s-a)$

b)

$\mathcal{L}\{f(t-a)\,u_a(t)\} = \int_0^\infty f(t-a)\, u_a(t)\, e^{-st}\, dt =$

$= \int_a^\infty f(t-a) e^{-st}\, dt = \int_0^\infty f(u) e^{-(u+a)s}\, du = e^{-as} \int_0^\infty f(u) e^{-su}\, du =$

$= e^{-as}\,\mathcal{L}\{f(u)\} = e^{-as} F(s)$

c) Say $g(t) = f(t-a) \Rightarrow g(t+a) = f(t)$ and (b) becomes,

$$\mathcal{L}\{g(t)\, u_a(t)\} = e^{-as}\, \mathcal{L}\{g(t+a)\}$$

since $g$ is arbitrary $g \to f$ gives the formula in ().

3)

a) $\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{\delta(t)\}$, $y(0) = y'(0) = 0$

$$\Rightarrow (as^2+bs+c)Y(s) = e^{-0s} = 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{as^2+bs+c}$$

if

$a=1,\ b=-2\ \ c=-3$

$$Y(s) = \frac{1}{s^2-2s-3} = \frac{1}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)}$$

$A = \frac{1}{2}\quad B = -\frac{1}{2}$

$$\Rightarrow Y(s) = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-3}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}c^{3t}$$

b) $Y(s) = \dfrac{1}{s^2+4s+4} = \dfrac{1}{(s+2)^2} \Rightarrow y(t) = te^{-2t}$

c) $Y(s) = \dfrac{1}{s^2-4s+13} = \dfrac{1}{(s-2)^2+9} = \dfrac{1}{3}\dfrac{3}{(s-2)^2+9}$

$\Rightarrow y(t) = \dfrac{1}{3}e^{2t}\sin(3t)$

d) $Y(s) = \dfrac{1}{s^2+9} = \dfrac{1}{3}\dfrac{1}{s^2+9} \Rightarrow y(t) = \dfrac{1}{3}\cos(3t)$

2) For $\omega = 1$

$Y(s) = \dfrac{s-1}{s^2+4} + \dfrac{3s}{s^2+1}\dfrac{1}{s^2+4}$ : $\dfrac{3s}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$\Rightarrow 3s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

Const : $0 = 4B + D$

$s$ : $3 = 4A + C$

$s^2$ : $0 = B + D$

$s^3$ : $0 = A + C$

$\Rightarrow B = D = 0$

$A = -C \Rightarrow 3 = 4A - A$

$\Rightarrow A = 1$

$C = -1$

a) $\Rightarrow y(t) = \cos(2t) - \dfrac{1}{2}\sin(2t) +$

$+ \cos(t) - \cos(2t)$

b) $Y(s) = \dfrac{s-1}{s^2+4} + \dfrac{3s}{(s^2+4)^2}$

$\Rightarrow y(t) = \cos(2t) - \dfrac{1}{2}\sin(2t) + \dfrac{3}{4}t\sin(2t)$

c) The oscillator in (a) has bounded oscillations in the long term while (b) does not. (b) is a Resonant case.

5)

$$Y(s) = \frac{m\alpha s + m\beta}{ms^2+k} + \frac{Ae^{-Ts}}{ms^2+k} =$$

$$= \frac{\alpha \cancel{m}}{\cancel{m}} \cdot \frac{s}{s^2+\omega^2} + \frac{\beta \cdot \cancel{m}}{\cancel{m}\,\omega} \frac{1 \cdot \omega}{s^2+\omega^2} + \frac{A}{m\omega} e^{-Ts} \frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow y(t) = \alpha \cos(\omega t) + \frac{\beta}{\omega} \sin(\omega t) + \frac{A}{m\omega} u_T(t) \sin(\omega(t-T)) =$$

$$= \left[\alpha - \frac{A}{m\omega} u_T(t) \sin(\omega T)\right] \cos(\omega t) + \left[\frac{\beta}{\omega} + \frac{A u_T(t) \cos(\omega T)}{m\omega}\right] \sin(\omega t)$$

for $t \geq T$ $u_T(t) = 1$ then set the following

$$\alpha - \frac{A}{m\omega} \sin(\omega T) = 0 \Rightarrow A = \frac{\alpha m \omega}{\sin(\omega T)}$$

and

$$\frac{\beta}{\omega} + \frac{A}{m\omega} \cos(\omega T) = \frac{\beta}{\omega} + \frac{\alpha m \omega}{m\omega} \frac{\cos(\omega T)}{\sin(\omega T)} = 0$$

$$\Rightarrow T = \frac{1}{\omega} \arctan\left(-\frac{\alpha\omega}{\beta}\right)$$