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Note Title

11/10/2006

Read 13 Boas

First 4 sections cover various examples in Cartesian coord.

Laplace' equation

$$\nabla^2 u = 0$$

Poisson's equation

$$\nabla^2 u = f(x, y, z)$$

Diffusion Equation

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}$$

wave equation (scalar)

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

Schrödinger's equation

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}$$

↑

ψ = wavefunction

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plucked string mathem.
example on wiki

Comment: why $\omega^2 - \omega^2$
and not $+\omega^2$?

suppose $\frac{\ddot{T}}{T} = +\omega^2$

then $\ddot{T} - \omega^2 T = 0$

Solution: $T = T_0 e^{-\omega t}$

where as

$$\ddot{T} + \omega^2 T = 0$$

Solution: $T = T e^{\pm i\omega t}$

So the physics dictates the sign of the separation constant

$-\omega^2 \Rightarrow$ oscillatory solutions

$+\omega^2 \Rightarrow$ expon. decaying

2D case in cartesian

$$c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}$$

$$u(x, y, t) = X(x) Y(y) T(t)$$

⋮

$$\ddot{T} = -\omega^2 T$$

$$c^2 \left(\frac{X''}{X} + \frac{Y''}{Y} \right) = -\omega^2$$

The time equation is the same as before.

$$\rightarrow \frac{X''}{X} + \frac{\omega^2}{c^2} = -\frac{Y''}{Y} = \text{const}$$

depends only
on X

depends only
on Y

$\frac{\omega^2}{c^2}$ could go on either

side

$$\left[\frac{\omega^2}{c^2} \right] = \frac{\left(\frac{1}{\text{time}} \right)^2}{\left(\text{dist} / \text{time} \right)^2}$$

$$= \left(\frac{1}{\text{distance}} \right)^2$$

$\frac{1}{\text{distance}}$ = wavenumber

wavenumber (k) is the spatial analog of frequency

$$x \leftrightarrow k \text{ as } t \leftrightarrow \omega$$

$$\frac{\bar{X}''}{\bar{X}} + \frac{\omega^2}{c^2} = -\frac{\bar{Y}''}{\bar{Y}} = k_y^2$$

\Rightarrow 2 ODE'S

$$\bar{Y}'' + k_y^2 \bar{Y} = 0$$

$$\bar{X}'' + \left(\frac{\omega^2}{c^2} - k_y^2 \right) \bar{X} = 0$$

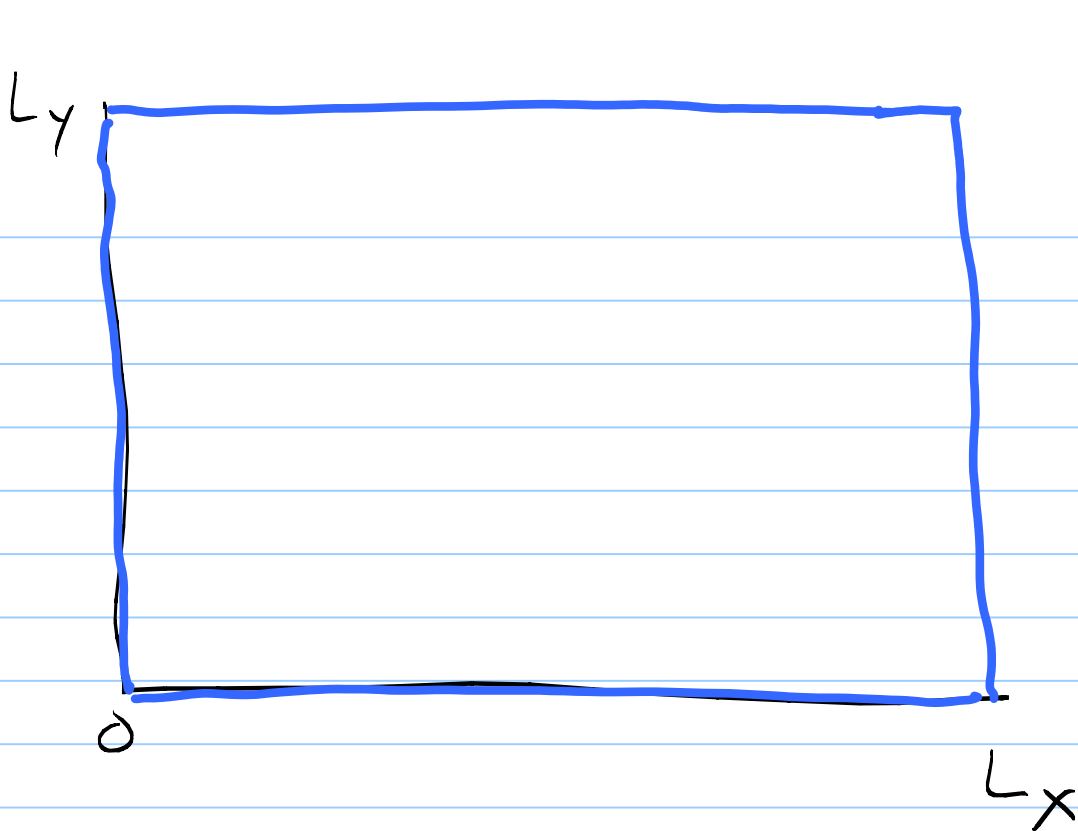
If we introduce a k_x too then we can let

$$\frac{\omega^2}{c^2} \equiv k^2 = k_x^2 + k_y^2,$$

so that we have

$$\nabla^2 X + k_x^2 X = 0$$

specific case, a rectangular drum (2d analog of a string)



$$u(x=0, y, t) = 0$$

$$u(x=L_x, y, t) = 0$$

$$u(x, y=0, t) = 0$$

$$u(x, y=L_y, t) = 0$$

} clamped
edges

So, as for the string

$$\bar{X}(x) = A \sin(k_x x)$$

$$\bar{Y}(y) = B \sin(k_y y)$$

and $k_x x = \frac{n \pi x}{L_x}$

$$k_y y = \frac{m \pi y}{L_y}$$

Since $\frac{c^2}{c^2} = k_x^2 + k_y^2$, we have

$$\frac{c^2}{c^2} = \left(\frac{n \pi}{L_x} \right)^2 + \left(\frac{m \pi}{L_y} \right)^2$$

or $\omega_{n,m}^2 = c^2 \pi^2 \left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2} \right)$

Spatial part of drum
solution = $\phi(x)\psi(y)$

$$\propto \sin\left(\frac{n\pi x}{L_x}\right) \sin\left(\frac{m\pi y}{L_y}\right)$$

$$\omega_{n,m}^2 = c^2 \pi^2 \left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2} \right)$$

temporal part given by

$$\propto e^{i\omega_{n,m} t}$$