Advanced Engineering Mathematics		**Homework Three**
Eigenproblems : Eigenvalues, Eigenvectors, Diagonalization, Self-Adjoint Operators		
Text: 8.1-8.4	Lecture Notes: 7-8	Lecture Slide: N/A

Quote of Homework Three Solutions	
Walk without rhythm, and it won't attract the worm. If you walk without rhythm, you never learn.	
	Norman Quentin Cook : Weapon of Choice (2000)

1. Eigenvalues and Eigenvectors

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

1.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

Recall that the associated eigenproblem for a square matrix $\mathbf{A}_{n\times n}$ is defined by $\mathbf{Ax} = \lambda\mathbf{x}$ whose solution is found via the following auxiliary equations:

- Characteristic Polynomial : $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
- Associated Null-space : $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

For each of the previous matrices we have:

$$\begin{aligned} \det(\mathbf{A}_1 - \lambda\mathbf{I}) &= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \end{aligned}$$

Case $\lambda_1 = 1$:

$$\left[\ \mathbf{A}_1 - \lambda_1\mathbf{I}\ \middle|\ \mathbf{0}\ \right] = \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} 3x_1 = -x_3 \\ -2x_1 = 0 \\ x_2 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

A basis for this eigenspace associated with $\lambda = 1$ is $B_{\lambda=1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Case $\lambda_2 = 2$:

$$\left[\ \mathbf{A}_1 - \lambda_2\mathbf{I}\ \middle|\ \mathbf{0}\ \right] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_3/2 \\ x_2 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} x_3$$

A basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Case $\lambda_3 = 3$:

$$\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_3 \mathbf{I} & \mathbf{0} \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow$$

$$\Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

A basis for this eigenspace is $B_{\lambda=3} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$$\det(\mathbf{A}_2 - \lambda \mathbf{I}) = (3-\lambda)(1-\lambda) - (-2) = \lambda^2 - 4\lambda + 5 \Rightarrow \lambda = \frac{-(-4) \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i$$

Case $\lambda = 2 \pm i$:

$$\left[\begin{array}{c|c} \mathbf{A}_2 - \lambda \mathbf{I} & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc|c} 3-(2\pm i) & 1 & 0 \\ -2 & 1-(2\pm i) & 0 \end{array} \right] =$$

$$= \left[\begin{array}{cc|c} 1 \mp i & 1 & 0 \\ -2 & -1 \mp i & 0 \end{array} \right].$$

Row-reduction with complex numbers is possible. However, it is easier to note that for a two-by-two system we can use either row, in this case the first,

(1) $$(1 \mp i)x_1 + 1x_2 = 0 \iff (1 \mp i)x_1 = -x_2,$$

to define the ratio between x_1 and x_2.[1] That is, if $x_1 = -1$ then $x_2 = 1 \mp i$ and thus the eigenvectors, like the eigenvalues, come in complex conjugate pairs $\mathbf{x} = [-1 \;\; 1 \mp i]^{\mathrm{T}}$.[2]

Since $\mathbf{A}_3$ is triangular we know the eigenvalues of$\mathbf{A}_3$ are,

$$\begin{array}{rcll} \lambda_1 & = & 4 & \text{(With algebraic multiplicity of 2)}, \\ \lambda_2 & = & 2 & \text{(With algebraic multiplicity of 2)}. \end{array}$$

Case $\lambda_1 = 4$:

$$\left[\begin{array}{c|c} \mathbf{A}_3 - \lambda_1 \mathbf{I} & \mathbf{0} \end{array} \right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array} \right] \Rightarrow \begin{array}{l} -2x_3 = 0 \\ x_1 = 2x_4 \\ x_2, x_4 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the basis for this eigenspace is $B_{\lambda=4} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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[1] This only works for two-by-two problems. In higher dimensions it is not possible to fix one variable and uniquely define the remaining variables.

[2] For real matrices, complex Eigenvalues and eigenvectors must occur in conjugate pairs.

Case $\lambda = 2$:

$$\left[\begin{array}{c|c} \mathbf{A}_3 - \lambda_2\mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3, x_4 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned} \det\left(\begin{bmatrix} .1-\lambda & .6 \\ .9 & .4-\lambda \end{bmatrix}\right) &= (.4-\lambda)(.1-\lambda) - .54 = \lambda^2 - .5\lambda - .54 + .04 = \\ &= \lambda^2 - .5\lambda - .5 \Rightarrow \lambda = \frac{-(-.5) \pm \sqrt{(-.5)^2 - 4(1)(-.5)}}{2(1)} = \frac{.5 \pm 1.5}{2} \Rightarrow \lambda_1 = 1,\ \lambda_2 = -.5 \end{aligned}$$

Case $\lambda_1 = 1$:

(2) $$[\mathbf{A}_4 - \lambda_1\mathbf{I}|\mathbf{0}] = \left[\begin{array}{cc|c} -.9 & .6 & 0 \\ .9 & .6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -.9 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Case $\lambda_2 = -.5$:

(3) $$[\mathbf{A}_4 - \lambda_2\mathbf{I}|\mathbf{0}] = \left[\begin{array}{cc|c} .6 & .6 & 0 \\ .9 & .9 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} .6 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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(4) $$\det(\mathbf{A}_5 - \lambda\mathbf{I}) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Case $\lambda = \pm 1$:

(5) $$[\mathbf{A}_5 - \lambda\mathbf{I}|\mathbf{0}] = \left[\begin{array}{cc|c} \mp 1 & -i & 0 \\ i & \mp 1 & 0 \end{array}\right] \Rightarrow \mp x_1 - ix_2 = 0 \iff \mp x_1 = ix_2 \Rightarrow \mathbf{x} = \begin{bmatrix} i \\ \mp 1 \end{bmatrix}$$

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2. Diagonalization

2.1. **Eigenbasis and Decoupled Linear Systems.** Find the diagonal matrix $\mathbf{D}_i$ and vector $\tilde{\mathbf{Y}}$ that completely decouples the system of linear differential equations $\frac{d\mathbf{Y}_i}{dt} = \mathbf{A}_i\mathbf{Y}_i$ for $i = 3, 4, 5$.

If one finds $n-$many eigenvectors for an $n \times n$ matrix then it is possible to find a diagonal matrix *similar* to $\mathbf{A}_{n\times n}$. That is, if $\mathbf{A}$ has $n-$many eigenvectors then $\mathbf{A}$ has the following diagonal decomposition,

(6) $$\mathbf{A} = \mathbf{PDP}^{-1},$$

where $\mathbf{D}$ is a diagonal matrix whose elements are eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is an invertible matrix whose columns are the eigenvectors corresponding to the eigenvalue elements of $\mathbf{D}$. This sort of decomposition is important because if we recall the coordinate changes described in 02.LS.Geometry in $\mathbb{R}^n$ then we can see that the eigenvector matrix defines a coordinate change for a given linear problem,

(7) $$\frac{d\mathbf{Y}_i}{dt} = \mathbf{A}_i\mathbf{Y}_i = \mathbf{P}_i\mathbf{D}_i\mathbf{P}_i^{-1}\mathbf{Y}_i \iff \frac{d\mathbf{P}_i^{-1}\mathbf{Y}_i}{dt} = \mathbf{D}_i\mathbf{P}_i^{-1}\mathbf{Y}_i \Rightarrow \frac{d\tilde{\mathbf{Y}}_i}{dt} = \mathbf{D}_i\tilde{\mathbf{Y}}_i,$$

where $\tilde{\mathbf{Y}}_i = \mathbf{P}_i^{-1}\mathbf{Y}_i$ for $i = 1,2,3,4,5$. We then say that $\tilde{\mathbf{Y}}_i$ is the coordinates of $\mathbf{Y}_i$ relative to the eigenvector basis. What is interesting is that the problem, under this coordinate system, has become,

$$(8) \qquad \frac{d\tilde{\mathbf{Y}}_i}{dt} = \mathbf{D}_i\tilde{\mathbf{Y}}_i \iff \frac{d}{dt}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \vdots \\ \tilde{y}_n \end{bmatrix} = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{bmatrix}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \vdots \\ \tilde{y}_n \end{bmatrix},$$

which is completely decoupled and therefore solvable without any row-reduction or eigen-methods. For each of the systems $i = 3,4,5$ we have the following:

$$(9) \qquad [\mathbf{P}_3|\mathbf{I}] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right] = \left[\mathbf{I}|\mathbf{P}_3^{-1}\right] \Rightarrow \tilde{\mathbf{Y}} = \mathbf{P}_3^{-1}\mathbf{Y} = \begin{bmatrix} y_2 \\ .5y_1 \\ y_3 \\ .5y_1 + y_4 \end{bmatrix}, \text{ and } \mathbf{D}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$(10) \qquad \mathbf{P}_4 = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \Rightarrow \mathbf{P}_4^{-1} = \begin{bmatrix} 1 & 1 \\ 3/5 & -2/5 \end{bmatrix}, \text{ and } \mathbf{D}_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$(11) \qquad \mathbf{P}_5 = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \Rightarrow \mathbf{P}_5^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{i}{2}\begin{bmatrix} -1 & i \\ -1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}, \text{ and } \mathbf{D}_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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3. Regular Stochastic Matrices

For the *regular stochastic matrix* $\mathbf{A}_4$, define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$.

3.1. **Limits of Time Series.** Show that $\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.

First, we note that we have already found the steady-state vector $\mathbf{q}$ since it is the eigenvector associated with $\lambda_1 = 1$. Now, the question is how to raise a matrix to an infinite power. Generally, it is unclear whether this processes converges and if it does, what it converges to. However, diagonalization offers us hope since,

$$(12) \qquad \lim_{n\to\infty} \mathbf{A}^n = \lim_{n\to\infty} \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \mathbf{P}\lim_{n\to\infty} \mathbf{D}^n\mathbf{P}^{-1},$$

where $[\mathbf{D}^n]_{ij} = d_{ii}^2\delta_{ij}$. Though calculating $\mathbf{A}^n$ is hard, calculating $\mathbf{D}^n$ is easy and more importantly, limiting processes on matrices now reduce to limiting processes on scalars, which is well-understood. In this case we have,

$$(13) \qquad \lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \lim_{n\to\infty} \mathbf{P}_4\mathbf{D}_4^n\mathbf{P}_4^{-1}\mathbf{x} = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix}\begin{bmatrix} \lim_{n\to\infty} 1^n & 0 \\ 0 & \lim_{n\to\infty} (-.5)^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3/5 & -2/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(14) \qquad = \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .4(x_1 + x_2) \\ .6(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \mathbf{q}.$$

4. Orthogonal Diagonalization and Spectral Decomposition

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ then their inner-product is defined to be $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$. Also, in this case, the 'length' of the vector is taken to be $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

4.1. **Self-Adjointness.** Show that $\mathbf{A}_5$ is a self-adjoint matrix.

First note that $\mathbf{A}_5 = \sigma_y$ from homework one. It was shown in this homework that $\mathbf{A}_5^{\mathrm{H}} = \mathbf{A}_5$.

4.2. **Orthogonal Eigenvectors.** Show that the eigenvectors of $\mathbf{A}_5$ are orthogonal with respect to the inner-product defined above.

Vectors are orthogonal if their inner-product is zero. With our previous definition of inner-product, the calculation,

$$\mathbf{x}_{\mp}^{\mathrm{H}}\mathbf{x}_{\pm} = [\bar{i} \;\; \mp 1]\begin{bmatrix} i \\ \mp 1 \end{bmatrix} = [-i \;\; \pm 1]\begin{bmatrix} i \\ \mp 1 \end{bmatrix} = -i \cdot i + \mp 1 \cdot \pm 1 = 1 - 1 = 0, \tag{15}$$

shows that the eigenvectors are orthogonal.

4.3. **Orthonormal Eigenbasis.** Using the previous definition for length of a vector and the eigenvectors of the self-adjoint matrix, construct an *orthonormal basis* for $\mathbb{C}^2$.

An orthonormal basis is an orthogonal basis where the basis vectors have all been scaled to have unit-length. Using our definition of inner-product to define a length we note,

$$\sqrt{\mathbf{x}_{\mp}^{\mathrm{H}}\mathbf{x}_{\mp}} = \sqrt{1+1} = \sqrt{2}, \tag{16}$$

which implies that the normalized eigenvectors are, $\mathbf{x}_{\mp} = [i\sqrt{2}/2 \quad \mp\sqrt{2}/2]^{\mathrm{T}}$.

4.4. **Orthogonal Diagonalization.** Show that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, where $\mathbf{U}$ is a matrix containing the normalized eigenvectors of $\mathbf{A}_5$.

We have seen from the previous problems that if you have enough eigenvectors then it is possible to find a diagonal decomposition for the matrix. Geometrically, this decomposition provides a natural coordinate system for which the solution to the associated linear problem is manifestly clear. This is a powerful result but it can be made stronger.

The general statement is,

- If a matrix is self-adjoint then it can always be diagonalized.[3] Moreover, eigenvectors associated with different eigenvalues are orthogonal to one another and the resulting matrix can be constructed to have the property $\mathbf{P}\mathbf{P}^{\mathrm{H}} = \mathbf{P}^{\mathrm{H}}\mathbf{P} = \mathbf{I}$.[4]

Since $\mathbf{A}_5$ is self-adjoint we an demonstrate this fact.

$$\mathbf{U} = \begin{bmatrix} i\frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \Rightarrow \mathbf{U}^{\mathrm{H}} = \begin{bmatrix} -i\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -i\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } \mathbf{U}\mathbf{U}^{\mathrm{H}} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}+\frac{1}{2} \end{bmatrix}. \tag{17}$$

Thus, $\mathbf{A}_5 = \mathbf{U}\mathbf{D}_5\mathbf{U}^{\mathrm{H}}$ where the decomposition has been found without using row-reduction to find an inverse matrix!

4.5. **Spectral Decomposition.** Show that $\mathbf{A}_5 = \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}$.

The previous result is quite powerful and can be used to derive other decompositions of the matrix $\mathbf{A}_5$. One such decomposition is called the spectral decomposition, which speaks to the action of $\mathbf{A}_5$ as a transformation. Assuming the given decomposition we consider the transformation,

$$\mathbf{A}_5\mathbf{y} = \left(\lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}\right)\mathbf{y} \tag{18}$$

$$= \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}}\mathbf{y} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}\mathbf{y} \tag{19}$$

$$= \lambda_1\langle\mathbf{x}_1, \mathbf{y}\rangle\mathbf{x}_1 + \lambda_2\langle\mathbf{x}_2, \mathbf{y}\rangle\mathbf{x}_2 \tag{20}$$

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[3]It can also be shown that its eigenvalues are always real. This is important to the theory of quantum mechanics where the eigenvalues are hypothetical measurements associated with a quantum system. It would be disconcerting if you stuck a thermometer into a quantum-turkey and it somehow read $3 + 2i$. Yikes!

[4]If the eigenvectors from a shared eigenspace are not orthogonal then it is possible to orthogonalize them by the *Gram-Schmidt* process.

which implies that $\mathbf{A}_5$ transforms the vector $\mathbf{y}$ by projecting this vector onto each eigenvector, rescaling it by a factor of λ_i and then linearly combines the results. To demonstrate this decomposition we calculate the following outer-product,

$$\mathbf{x}_{\mp}\mathbf{x}_{\mp}^{\mathrm{H}} = \begin{bmatrix} i\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \bar{i}\sqrt{2}/2 & \mp\sqrt{2}/2 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} i\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -i\sqrt{2}/2 & \mp\sqrt{2}/2 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} 1/2 & \mp i/2 \\ \pm i/2 & 1/2 \end{bmatrix}, \tag{23}$$

which gives,

$$\mathbf{A}_5 = 1 \cdot \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{24}$$

5. Introduction of Self-Adjoint Operators

Let L be a linear transformation defined by,

$$Lu = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right), \tag{25}$$

where the unknown function u must satisfy the boundary conditions,

$$k_1u(a) + k_2u'(b) = 0 \tag{26}$$

$$l_1u(b) + l_2u'(b) = 0. \tag{27}$$

Finding all nontrivial eigenfunctions of (25), which satisfy (26)-(27) is called a the *Sturm-Liouville (SL) Problem.*

5.1. **Equations in SL Form.** Let $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $k_1 = l_1 = 0$, $k_2 = l_2 = 1$ and $a = 0$, $b = \pi$. Show that the eigenvalue/eigenfunction pairs to the SLP are defined by $u_n(x) = c_n \cos\left(\sqrt{\lambda_n}x\right)$, $c_n \in \mathbb{R}$, $\lambda_n = n^2$, for $n = 0, 1, 2, 3, \ldots$.

The previous parameters define the following boundary value problem,

$$u'' + \lambda u = 0,\ \lambda \in \mathbb{R} \tag{28}$$

$$u'(0) = 0,\ u'(\pi) = 0, \tag{29}$$

whose ODE was solved, generally, in the previous homework assignment. Notice now that the boundary conditions imply that the solution must have a horizontal tangent-line at the right and left endpoints. Of the six functions that solve the ODE only two will satisfy this geometric condition. These functions are the cosine function and the constant function. Applying the right boundary condition to the cosine function gives,

$$u'(\pi) = c_1 \sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda}\pi = n\pi,\ n = 1, 2, 3, \ldots, \tag{30}$$

and since the constant is arbitrary we have that $u_n(x) = c_n \cos(\sqrt{\lambda_n}x) = c_n \cos(nx)$, where $n = 0, 1, 2, 3, \ldots$. Notice that we have also included $n = 0$ in the sequence, which gives the constant solution $u_0(x) = c_0 cos(0 \cdot x) = c_0$.

5.2. **Orthogonality of Eigenfunctions.** Using the abstract inner-product defined in homework 2 problem 5.2, $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$, show that the previous eigenfunctions form an orthogonal set. That is, show that $\langle u_n, u_m\rangle = \pi\delta_{nm}$ for $n = 1, 2, 3, \ldots$, and $m = 1, 2, 3, \ldots$.

If we recall problem 5 from the last homework we notice that a similar relation was shown for sine functions and that this integral was proven using exponential functions with imaginary argument. Though using these functions makes the integrals trivial it requires some amount of cleverness and it is natural to question why argue in this way. Well, if you go back to that argument and argue the same way but on the cosine functions you will find that, $\langle \cos(nx), \cos(mx)\rangle = \pi\delta_{nm}$, for $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$, which is our desired result. Thus, the one argument supplies both orthogonality relations! We summarize the logic of the last two problems:

- A self-adjoint matrix gives rise to an eigenvector coordinate system, which is naturally orthogonal. The idea of a matrix can be thought of as a linear transformation on a finite dimensional vector-space and this idea can be abstracted. In this abstraction the differential operator defined by (25) coupled to (26)-(27) defines a self-adjoint linear transformation and 5.2 shows that its 'eigenvectors' are also orthogonal. Since the SLP provides infinitely-many of these eigenvectors, it must be a self-adjoint linear transformation on an infinite dimensional vector-space.