MATH 332 - Linear Algebra July 2, 2009

Homework 5, Summer 2009 **Due**: July 6, 2009

Determinants - Row-Reductions - Properties - Inverse Matrices - Volumes

1. Given the following for matrices:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}.$$

Calculate the determinants of the previous matrices by theorem 2.2.4. In each case, state the row-operation on **A** and describe how it effects the determinant.

2. The following questions illustrate some important properties of the determinant.

(a) The determinant is not, in general, a linear mapping. That is, det: $\mathbb{R}^{n\times n} \to \mathbb{R}$ is **not**, in general, such that, $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A})+\det(\mathbf{B})$. The determinant is, in general, *multilinear*.[1] Show this for the domain $\mathbb{R}^{3\times 3}$ by verifying that $\det(\mathbf{A}) = \det(\mathbf{B}) + \det(\mathbf{C})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given as,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & u_1+v_1 \\ a_{21} & a_{22} & u_2+v_2 \\ a_{31} & a_{32} & u_3+v_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}.$$

(b) Show that if **A** is invertible, then $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.

(c) Let **A** and **P** be square matrices such that $\mathbf{P}^{-1}$ exists. Show that $\det(\mathbf{PAP}^{-1}) = \det(\mathbf{A})$.

(d) Let **U** be a square matrix such that $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$. Show that $\det(\mathbf{U}) = \pm 1$.

(e) Find a formula for $\det(r\mathbf{A})$ where $\mathbf{A} \in \mathbb{R}n \times n$ and $r \in \mathbb{R}$.

3. Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}. \tag{1}$$

(a) Find $\det(\mathbf{A})$ using cofactor expansion.

(b) Find $\det(\mathbf{A})$ using row reduction to echelon form.

4. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Calculate the adjugate of **A** and using theorem 3.3.8 calculate $\mathbf{A}^{-1}$.

5. The determinant has a geometric interpretation. In $\mathbb{R}^2$, $\det(\mathbf{A})$ is the area of the parallelogram formed by the two vectors $\mathbf{a}_1, \mathbf{a}_2$, where $\mathbf{A} = [\mathbf{a}_1\, \mathbf{a}_2]$. In $\mathbb{R}^3$, $\det(\mathbf{A})$ is the volume of the parallelepiped formed by the three vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where $\mathbf{A} = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3]$.

Using the concept of volume, explain why the determinant of a 3×3 matrix **A** is zero if and only if **A** is not invertable.

Note: No credit will be given for the use of Theorem 3.2.4. Also, note that there are two proofs here. The forward proof should assume that $\det(\mathbf{A}) = 0$ and conclude that **A** is singular by using the geometry formed by the column vectors. The backward proof should start assuming **A** is singular and conclude that the parallelepiped volume is zero. The two proofs together prove the *if and only if* statement above.

Hint: Use the invertible matrix theorem of 2.3 and a geometric description of linearly dependent vectors in $\mathbb{R}^3$.

[1] A multilinear map is a mathematical function of several vector variables that is linear in each variable. That is, if all columns except one are fixed, then the determinant is a linear function of that one column. See `http://en.wikipedia.org/wiki/Multilinear_map` for more information.