2_11_08

Note Title 2/10/2008

a) wavepackets + Dispersion

b) δ potential

Plane wave $\Psi(x,t) = e^{i(kx-\omega t)}$

The dependence of ω on k is called dispersion

Consider simultaneous prop. in $+x$ direction of 2 plane waves.

$$\Psi(x,t) = e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}$$

Here is a very simple model in which

1) $\omega = f(k)$

2) $k_1 = k - \epsilon$, $k_2 = k + \epsilon$ $\quad \epsilon \ll k$

So $\omega_1 = f(k_1) = f(k-\epsilon)$

$\approx f(k) - \epsilon \frac{df}{dk} = \omega - \epsilon U$

$\uparrow df/dk$

$\omega_2 = f(k_2) = f(k+\epsilon)$

$\approx f(k) + \epsilon \frac{df}{dk}$

$= \omega + \epsilon U$

$$\Psi(x,t) = e^{i(k_1 x - \omega_1 t)} + e^{i(k_2 x - \omega_2 t)}$$

$$= e^{i[(k-\epsilon)x - (\omega - \epsilon U)t]} + e^{i[(k+\epsilon)x - (\omega+\epsilon U)t]}$$

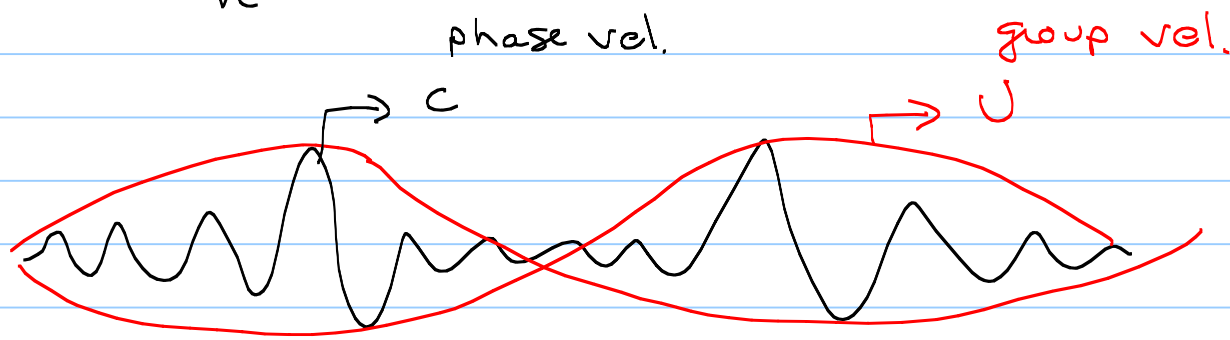
$$= e^{i(kx-\omega t)}\left[e^{-i\epsilon(x-Ut)} + e^{i\epsilon(x-Ut)}\right]$$

$$= e^{ik(x-\frac{\omega}{k}t)}\left[2\cos(\epsilon(x-Ut))\right]$$

↖ modulation

ω = carrier freq.

$$\frac{\omega}{k} = c$$



Now, instead of 2 frequencies ω use an infinite #.

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \omega(k)t)}\, dk \quad 🚩$$

Suppose that $\phi(k)$ is peaked around some particular value of k, k_0.
Then using

$$\omega(k) \simeq \underbrace{\omega(k_0)}_{\equiv\ \omega_0} + \frac{d\omega}{dk}(k-k_0)$$

inserting into 🚩 we have

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \omega_0 t - \omega'(k-k_0)t)}$$

Since ϕ is peaked around k_0 we should shift the integral to be centered around k_0

$$k = k_0 + s \longrightarrow \text{new variable}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(k_0+s)\, e^{i[(k_0+s)x - (\omega_0 + \omega' s)t]}\, ds$$

NB at $t=0$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int \phi(k_0+s)\, e^{i(k_0+s)x}\, ds$$

What we expect is that for $t \neq 0$ we will get the same result but x will be shifted

$$x \to x - \omega' t$$

So

$$\Psi(x,t) \approx \frac{1}{\sqrt{2\pi}} \int \phi(k_0+s)\, e^{i(k_0+s)(x-\omega' t)}\, e^{-i\omega_0 t}\, e^{ik_0\omega' t}\, ds$$

$$\underbrace{e^{-i(\omega_0 - k_0\omega')t}}_{} \frac{1}{\sqrt{2\pi}} \int \phi(k_0+s)\, e^{i(k_0+s)(x-\omega' t)}\, ds$$

this phase factor will not affect $|\Psi(x,t)|^2$.

$$\Psi(x,t) = e^{i(\ \ )} \underbrace{\Psi(x-\omega' t, 0)}_{}$$

This is essentially the definition of a "group"

$$V_g = \frac{d\omega}{dk} \qquad V_p = \frac{\omega}{k}$$

Remember, for free particle

$$\omega = \hbar k^2/2m$$

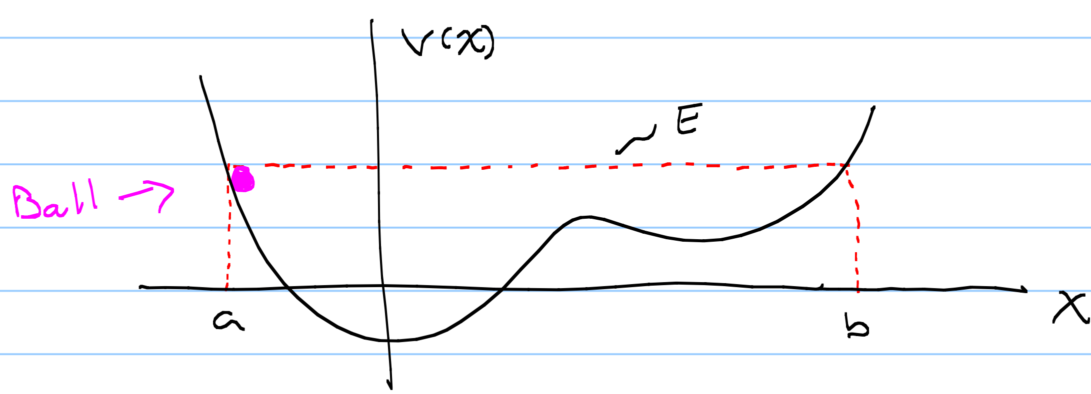
$$V_p = \frac{\hbar k}{2m} \qquad V_g = \frac{\hbar k}{m}$$

Bound states vs Scattering states.

- eigenstates of the ∞ potential well were rigorously zero outside the well
- For the QHO, the potential increased to ∞ as $x \to \infty$. This means that the QHO states cannot escape the clutches of the HO potential

These are all Bound States

Consider the classical potential



Start ball rolling with zero speed.
All its energy is potential

At $t=0$ $\quad E = V(a)$

as $t>0$ it picks up speed

$$E = V(x) + KE(x)$$

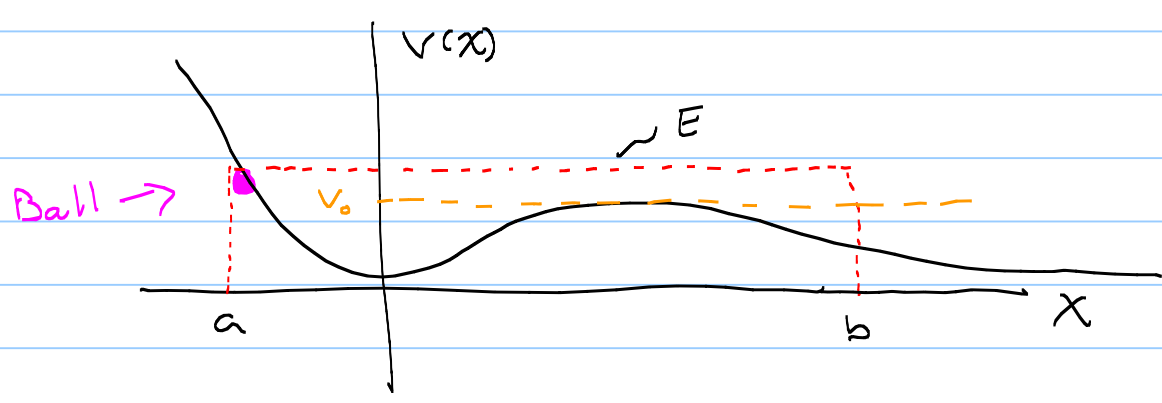
at some time later $x=b$ and the ball has lost all its KE

$$E = V(b)$$

it turns around and goes back to $x=a$, stops and repeats.

Assuming no friction

The points $x=a$, $x=b$ are the classical turning points.



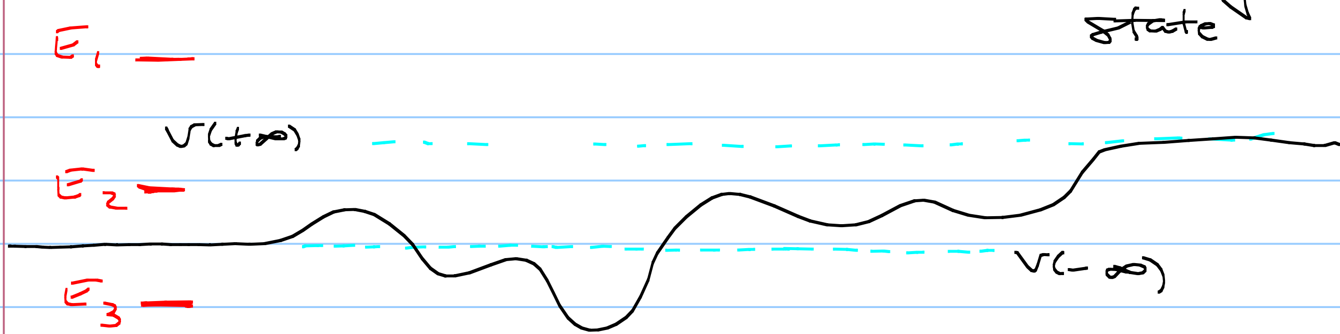
Now a is a turning point but if the energy of the ball is greater than V_0, the ball will continue onto the right.

This is a scattering state.

So, you initialing the particle with some energy E.

$E < V(-\infty)$ and $E < V(+\infty) \Rightarrow$ Bound State

$E > V(-\infty)$ or $E > V(+\infty) \Rightarrow$ Scattering state

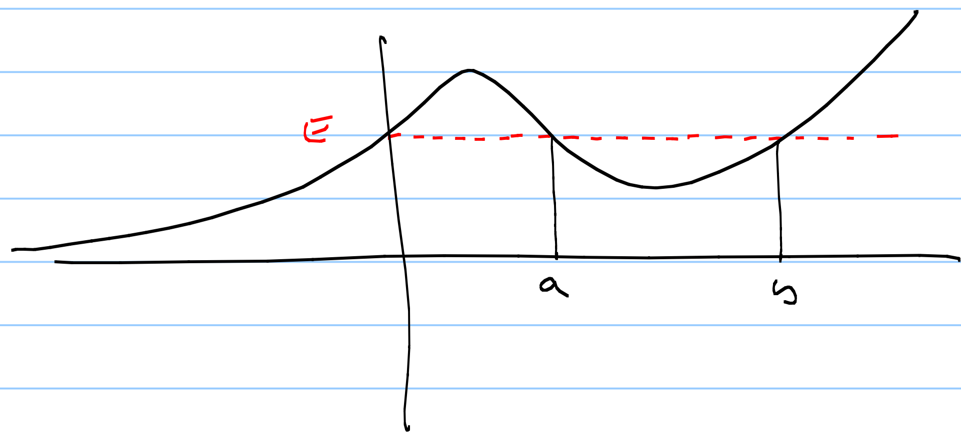


E_1 scattering state
E_2 Scattering state
E_3 bound state

usually $V(+\infty) = V(-\infty) = 0$

So in this case $E<0$ bound state

$E>0$ scattering state.



a, b are classical turning points for energy E but since $E > V(-\infty)$ this is a quantum scattering state.

Delta function well: $V(x) = -\alpha\,\delta(x)$

obviously this is artificial but it's a nice place to start.

$$H\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\psi = E\psi.$$

well Break this up into 2 stages

1) x not close to zero

2) x close to zero

1) if $x \neq 0$ $\quad V(x) = 0$

$H\psi = E\psi$ reduces to

$$\frac{-\hbar^2}{2m}\psi''(x) = E\psi(x)$$

$$\psi''(x) = \frac{\sqrt{-2mE}}{\hbar} = \kappa$$

Lets first look at bound states so $E < 0$ and κ is real

$$\psi(x) = A e^{-\kappa x} + B e^{+\kappa x}$$

in the region $x < 0$ $\quad A = 0$ else the solution would blow up.

in the region $x > 0$

$$\psi(x) = F e^{-\kappa x} + G e^{+\kappa x}$$

$G = 0$

in order that $\psi(x)$ be continuous at $x=0$

$$\psi(x) = \begin{cases} B e^{kx} & x < 0 \\ B e^{-kx} & x > 0 \end{cases}$$

i.e $\quad \psi(x) = B e^{-k|x|}$