MATH 348 - Advanced Engineering Mathematics January 12, 2009
Homework 1, Spring 2009 **Due**: January 19, 2009

Matrix Algebra - Row Reduction - Solutions to Linear Systems

1. We known, by counterexamples, that matrix multiplication is a non-commutative binary operation. We define a commutator as a function, which takes in two matrices and returns one and is, in some sense, a measure of the binary operations lack of commutativity. We define the commutator and anti-commutation functions on matrices as,

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \tag{1}$$

The following matrices are the so-called Pauli spin matrices and have interesting commutation and anti-commutation relations and gives us fine setting to practice our matrix algebra. [1]

$$\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

Using the previous definitions show the following:

(a) $\sigma_i^2 = \mathbf{I}$ for $i = 1, 2, 3$. [2]

(b) $[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. [3]

(c) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. [4]

2. Given the linear system

$$\begin{aligned} 6x_1 + 18x_2 - 4x_3 &= 20 \\ -x_1 - 3x_2 + 8x_3 &= 4 \\ 5x_1 + 15x_2 - 9x_3 &= 11. \end{aligned}$$

Determine the general solution set to the linear system and describe this set geometrically. [5]

[1] The Pauli spin matrices are a set of Hermitian matrices, which are *unitary*. They have found several uses including describing strong interaction symmetries in particle physics and representing logic gates in quantum information theory.

[2] This statement encapsulates both the symmetric unitary properties of the matrices.

[3] Here we are using the so-called Levi-Civita symbol. This symbol is used to encode the following commonly encountered information,

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1) \text{ or } (3, 1, 2), \\ -1, & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2) \text{ or } (2, 1, 3), \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases} \tag{3}$$

[4] Here we use the so-called Kronecker delta function, which encodes the, also common, information,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases} \tag{4}$$

[5] Another way to ask this: 'Are there a set of points in $\mathbb{R}^3$ where the three previous planes intersect one another? If so, then what geometric object do the collection of these points form?' I hope that it is clear that if there a solution then these points could only form a point, line, or plane, depending on the number of free-variables you find by row-reduction.

3. Given the following augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right].$$

Determine h and k such that the corresponding linear system: [6]

(a) Is inconsistent.

(b) Is consistent with infinitely many solutions.

(c) Is consistent with a unique solution.

4. Suppose $a, b, c,$ and d are constants such the system

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned}$$

with $a \neq 0$ is consistent for all possible values of f and g. Using row reduction solve for x_1 and x_2 and list any constraints needed, on $a, b, c, d,$ for unique solutions. [7]

5. Given the matrix $\mathbf{A}$ and the vector $\mathbf{b}$.

$$\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}$$

Are there constants x_1 and x_2 such that $\mathbf{b}$ can be formed as a linear combination of the columns of $\mathbf{A}$? If so then what are they?[8]

[6] Hint: You will not need to find the reduced row-echelon form. Only the row-echelon form is needed.

[7] What we are trying to do here is find conditions on the coefficients a, b, c, d that will guarantee a single solution to the system. Remember that in 1-D we require that to have a unique solution to, $ax = 0$, a must be different from zero.

[8] Another way of asking this: 'Is $\mathbf{b}$ a *linear combination* of the columns of $\mathbf{A}$?'