Homework 2 due 8 Sept. 2006 in class
PH462 EM Waves and Optical Physics posted: 2 September 2005

1. The case of wave propagation in cylindrical symmetry is important for beam propagation both in free space and in waveguides (such as optical fibers).
   a. Start with the scalar wave equation $\nabla^2\psi = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$, and assume azimuthal symmetry (not true in general) so that $\psi = \psi(r, z, t)$. Write the wave equation in cylindrical coordinates, and use the method of separation of variables to show that $\psi(r, z, t) = f(r)\exp[i(k_z z \pm k_0 \mathrm{v}t)]$.
   b. Show that the radial wave equation is now
$$\frac{d^2E}{dr^2} + \frac{1}{r}\frac{dE}{dr} + \left(k_0^2 - k_z^2\right)E = 0 .$$
This is the 0^{th} order Bessel equation.
   c. Show that a solution to the equation you found in part a is given by $E(r) \propto J_0(kr)$, where J_0 is the 0^{th} order Bessel function.

Hints: First define a variable $u = kr$ to rewrite the equation in dimensionless form. Then put $J_0(u)$ into the new equation and show that it is a solution. To do the derivatives, you will need to make use of the derivative relations (called the recurrence relations) for the Bessel functions:
$J_\nu{}'(u) = J_{\nu-1}(u) - \frac{\nu}{u}J_\nu(u)$ and $J_\nu{}'(u) = -J_{\nu+1}(u) + \frac{\nu}{u}J_\nu(u)$.
Here $J_\nu(u)$ is the Bessel function of order ν. Note that for the special case $\nu=0$, the second relation reduces to $J_0{}'(u) = -J_1(u)$. These recurrence relations are the equivalent of, for example, $(d/du)\cos u = -\sin u$ for the Bessel functions. Don't use the series expansions of the functions. There is a second solution, $Y_\nu(u)$, using Abramowitz and Stegun notation, that diverges at the origin. The $J_\nu(u)$ and $Y_\nu(u)$ solutions correspond to *standing* waves. The superpositions $H_\nu^{(1)} = J_\nu(u) + iY_\nu(u)$ and $H_\nu^{(2)} = J_\nu(u) - iY_\nu(u)$ are traveling wave solutions. This is analogous to sin(*u*) and cos(*u*) as standing waves and exp (*i u*) = cos(*u*)+*i* sin(*u*) as a traveling wave.

   d. A cylindrical wave is often approximated as $\cos kr/\sqrt{r}$ for large *kr*. Make a plot of both $J_0(kr)$ and $\cos kr/\sqrt{r}$ to show that they are the same at sufficiently large *kr*. Adjust the peak amplitude for a good comparison. The phase shift between the two functions should be π/4. In Mathematica, $J_0(kr)$ is written as BesselJ[0, k r]
2. Consider an electromagnetic plane wave, with field amplitude E_0 traveling in vacuum with a wavevector magnitude k_0 at an angle of +15^o to the *z*-axis. The *k*-vector is in the *x*-*z* plane and the *E*-field direction is along the *y*-axis.
   a. Write an expression in Cartesian coordinates for the vector electric field of this wave, writing the functional dependence in terms of the cosine function.
   b. Use one of the Maxwell equations to calculate the *B*-field for the wave in part a.

3. Describe the polarization state of a wave with the Jones vector $\begin{pmatrix} -i \\ 2 \end{pmatrix}$. Write the Jones vector that is orthogonal to this vector and describe its polarization state. When using complex notation for polarization states, the dot product of two vectors $\mathbf{E_1}$ and $\mathbf{E_2}$ is calculated as $\mathbf{E}_1^* \cdot \mathbf{E_2}$
4. Demonstrate using the Jones vector notation that right- and left-circularly polarized light waves are orthogonal.
5. A beam is polarized in the vertical (y) direction, and propagates in the z-direction. It passes through a half wave plate which can be rotated around the z-axis. Suppose the ordinary axis is at an angle θ to the x-axis. Express the input state as a linear combination of linearly polarized basis vectors that are aligned with the crystal axes. Using this representation of the input wave, apply the relative propagation phase shift to the wave that results from the half-wave plate. Finally, express the output wave in terms of the original basis (the x-y coordinate system) and show that the waveplate rotates linear polarization by 2θ.