<u>15</u>

Self-phase modulation
Fourier split step pulse propagation
temporal solitons

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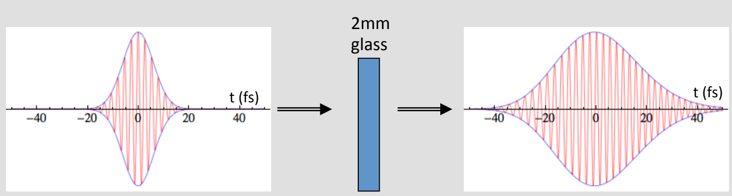
Pulse propagation: t/ω domain

- Dispersion in a system will stretch a short pulse:

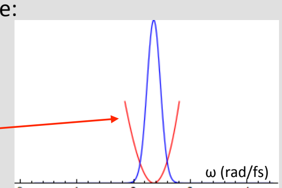


- Linear propagation is best represented in ω space:

$$E_{out}(\omega) = A(\omega - \omega_0)e^{i\phi(\omega)}$$

Spectral phase

$$\phi(\omega) = kL = \frac{\omega}{c}n(\omega)L$$

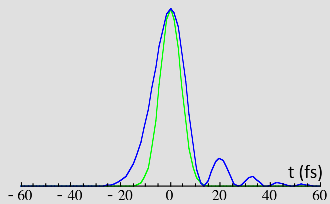


- Expand φ(ω) in series:

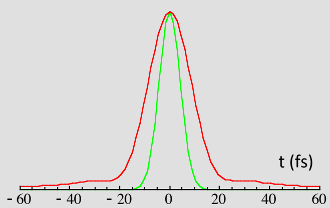
$$\phi(\omega) = \phi_0 + \phi_1(\omega - \omega_0) + \tfrac{1}{2}\phi_2(\omega - \omega_0)^2 + \tfrac{1}{3!}\phi_3(\omega - \omega_0)^3 + \tfrac{1}{4!}\phi_4(\omega - \omega_0)^4 + \cdots$$

Effects of residual high-order phase

- Compensate linear chirp, φ_2, only (φ_3-limited) :



- Compensate φ_2 and φ_3 (φ_4-limited):



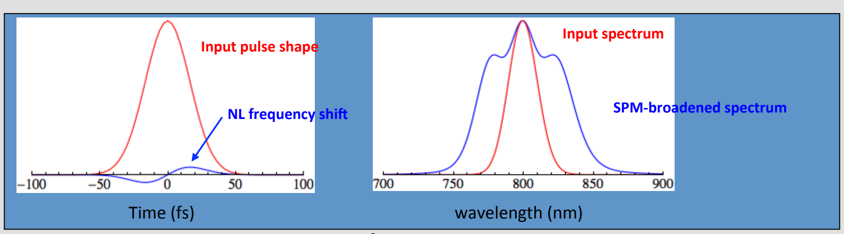
Nonlinear phase shifts: self-phase modulation

At high intensity, the refractive index can be changed noticeably:

$$-i\frac{dA}{dz} = \underbrace{\gamma_0 |A|^2 A}_{\text{SPM}} + \beta_2 \frac{\partial^2 A}{\partial z^2}$$

Typically, relative sign of γ_0 and β_2 do not allow for solitons.

B-integral (NL phase shift): $B \approx \gamma_0 |A|^2 L \longrightarrow L_{NL} \approx 1/\gamma_0 |A|^2$



Dispersion length: $L_D \sim \frac{\tau_p^2}{\beta_2}$

Output spectrum and pulse shape are dramatically affected by dispersion.

Self-phase modulation & continuum generation

The self-phase-modulated pulse develops a phase vs. time proportional to the input pulse intensity vs. time.

Pulse intensity vs. time

$$E_0(z,t) = E_0(0,t)\exp\left[i\,k_0\,n_2\,I(t)\,z\right]$$

The further the pulse travels, the more modulation occurs.

That is: $\boxed{\phi(z,t) \approx k_0\,n_2\,I(t)\,z}$

A flat phase vs. time yields the narrowest spectrum. If we assume the pulse starts with a flat phase, then SPM broadens the spectrum.

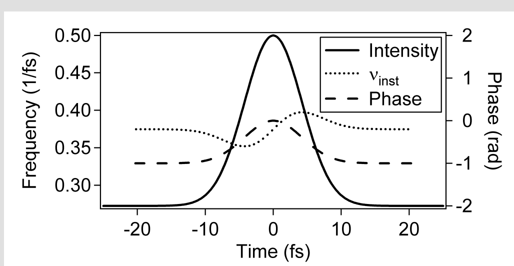
This is not a small effect! A total phase variation of hundreds can occur!
A broad spectrum generated in this manner is called Continuum.

The instantaneous frequency vs. time in SPM

$$\phi(z,t) \approx k_0\,z\,n_2\,I(t)$$

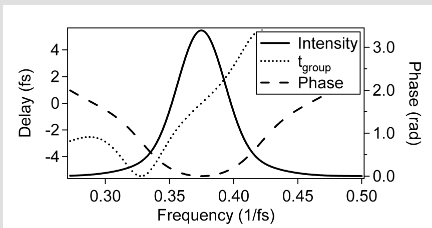
$$\omega_{inst}(t) = -\frac{\partial\phi(z,t)}{\partial t} = -k_0\,z\,n_2\,\frac{\partial I(t)}{\partial t}$$

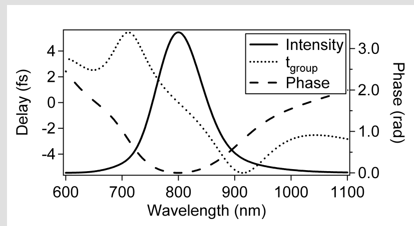
A 10-fs, 800-nm pulse that's experienced self-phase modulation with a peak magnitude of 1 radian.



Self-phase-modulated pulse in the frequency domain

The same 10-fs, 800-nm pulse that's experienced self-phase modulation with a peak magnitude of 1 radian.

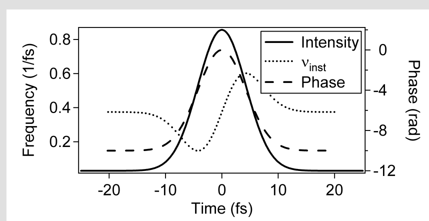


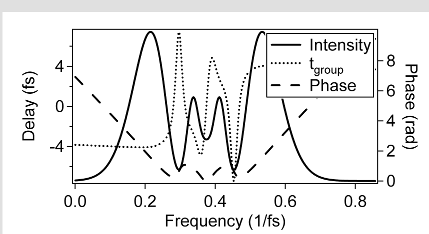


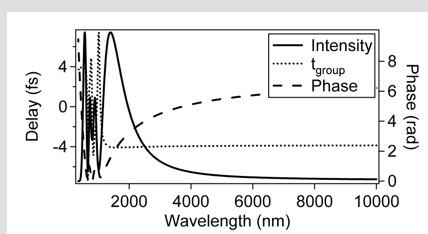
It's easy to achieve many radians for phase delay, however.

A highly self-phase-modulated pulse

A 10-fs, 800-nm pulse that's experienced self-phase modulation with a peak magnitude of 10 radians







Note that the spectrum has broadened significantly. When SPM is very strong, it broadens the spectrum a lot. We call this effect continuum generation.

Experimental continuum spectrum in a fiber

Continua created by propagating 500-fs 625nm pulses through 30 cm of single-mode fiber.

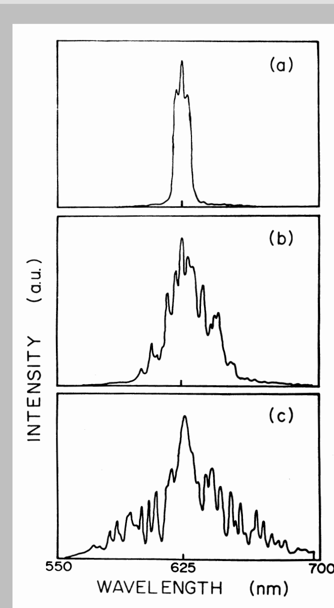
The Supercontinuum Laser Source, Alfano, ed.

Broadest spectrum occurs for highest energy.

Low Energy

Medium Energy

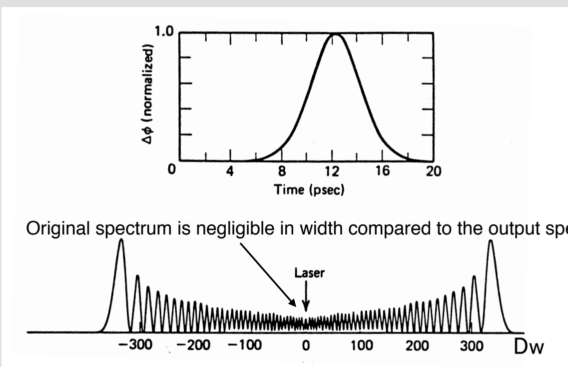
High Energy



Continuum generation simulations

Instantaneously responding n_2; maximum SPM phase = 72π radians

Input Intensity vs. time (and hence output phase vs. time)



Output spectrum:

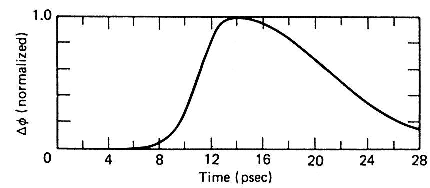
The Super-continuum Laser Source, Alfano, ed.

Oscillations occur in spectrum because all frequencies occur twice and interfere, except for inflection points, which yield maximum and minimum frequencies.

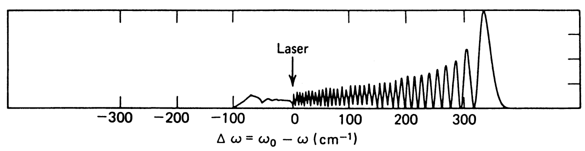
Continuum generation simulation

Noninstantaneously responding n_2; maximum SPM phase = 72π radians

Output phase vs. time (≠ input intensity vs. time, due to slow response)



Output spectrum:

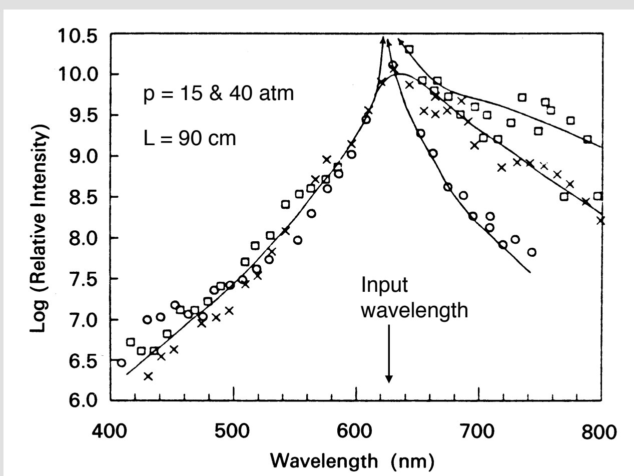


Asymmetry in phase vs. time yields asymmetry in spectrum.

The Supercontinuum Laser Source, Alfano, ed.

Experimental continuum spectra

625-nm (70 fs and 2 ps) pulses in Xe gas

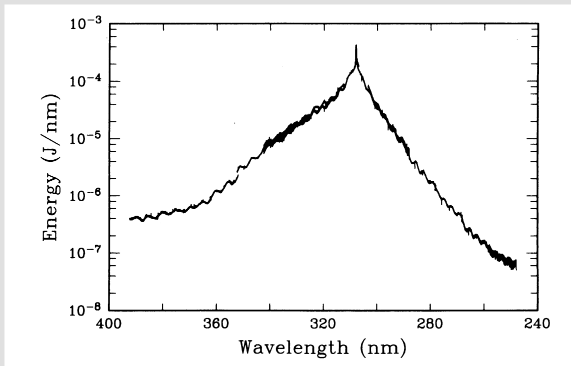


The Supercontinuum Laser Source, Alfano, ed.

Data taken by Corkum, et al.

Ultraviolet continuum

4-mJ 160-fs 308-nm pulses in 40 atm of Ar; 60-cm long cell.



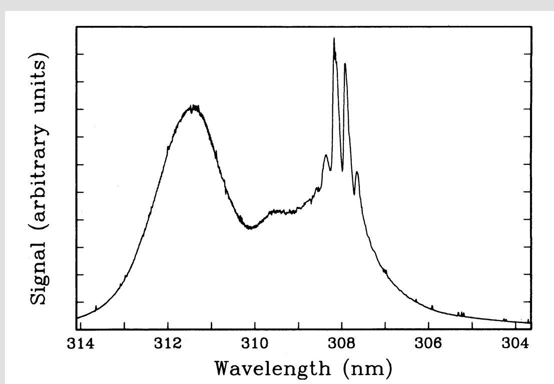
Lens focal length = 50 cm.

Good quality output mode.

The Supercontinuum Laser Source, Alfano, ed.

UV Continuum in Air!

308 nm input pulse; weak focusing with a 1m lens.

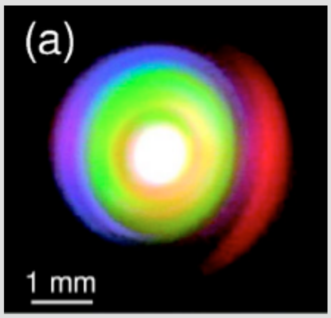


The Super-continuum Laser Source, Alfano, ed.

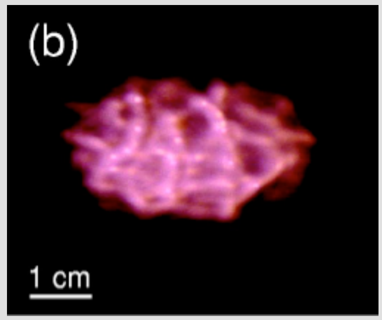
Continuum is limited when GVD causes the pulse to spread, reducing the intensity.

Continuum in air

Use a negatively chirped pulse to pre-compensate for the dispersion of air. Then "temporal focusing" occurs. The self-focusing can compensate diffraction: light bullets! Usually this only happens for pieces of the beam: filamentation, and it's messy.



Conical emission from a fs beam in air, near the critical power P_{cr}.



Beam profile of a high-power beam (~1000P_{cr}) after 15m. Note the multiple filamentation.

Continuum Generation: Good news and bad news

Good news:

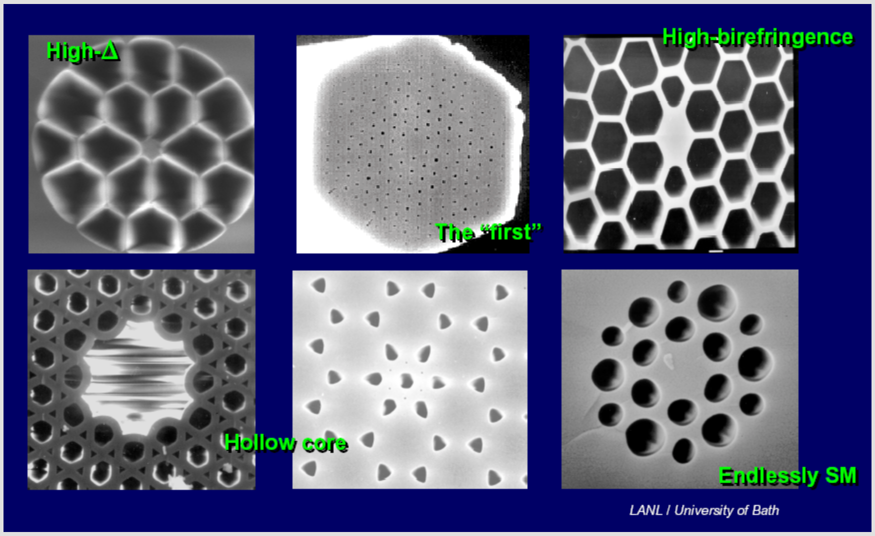
It broadens the spectrum, offering a useful ultrafast white-light source and possible pulse shortening.

Bad news:

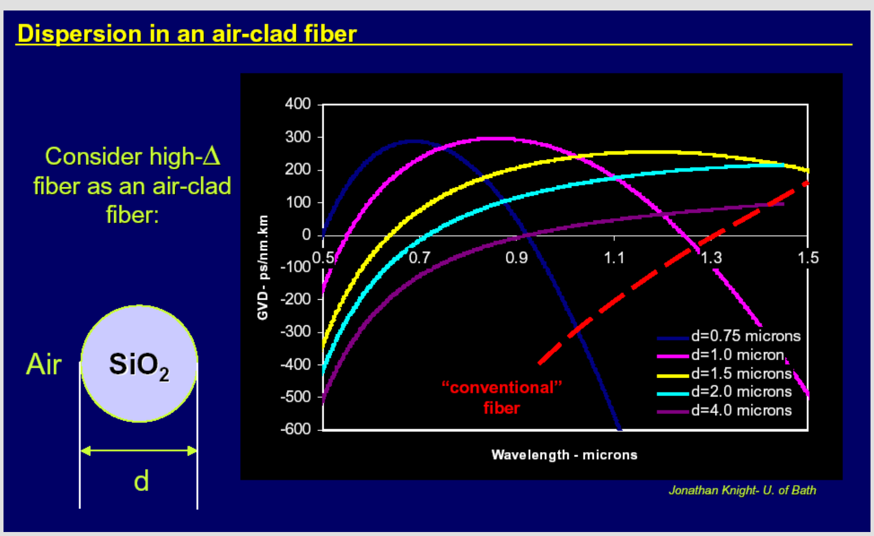
Pulse shapes are uncontrollable.
Theory is struggling to keep up with experiments.
In a bulk medium, continuum can be high-energy, but it's a mess spatially.
In a fiber, continuum is clean, but it's low-energy.
In hollow fibers, things get somewhat better.

Main problem: dispersion spreads the pulse, limiting the spectral broadening.

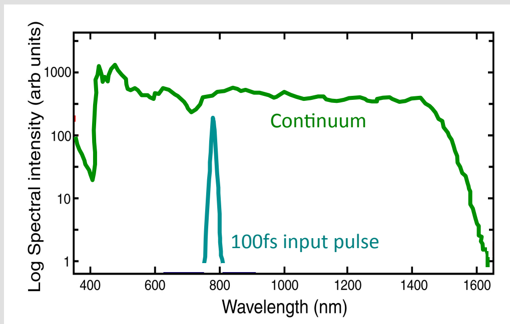
Microstructure optical fiber

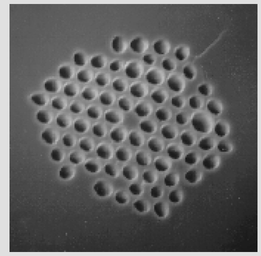


Microstructure optical fibers modify dispersion.



The continuum from microstructure optical fiber is ultrabroadband.



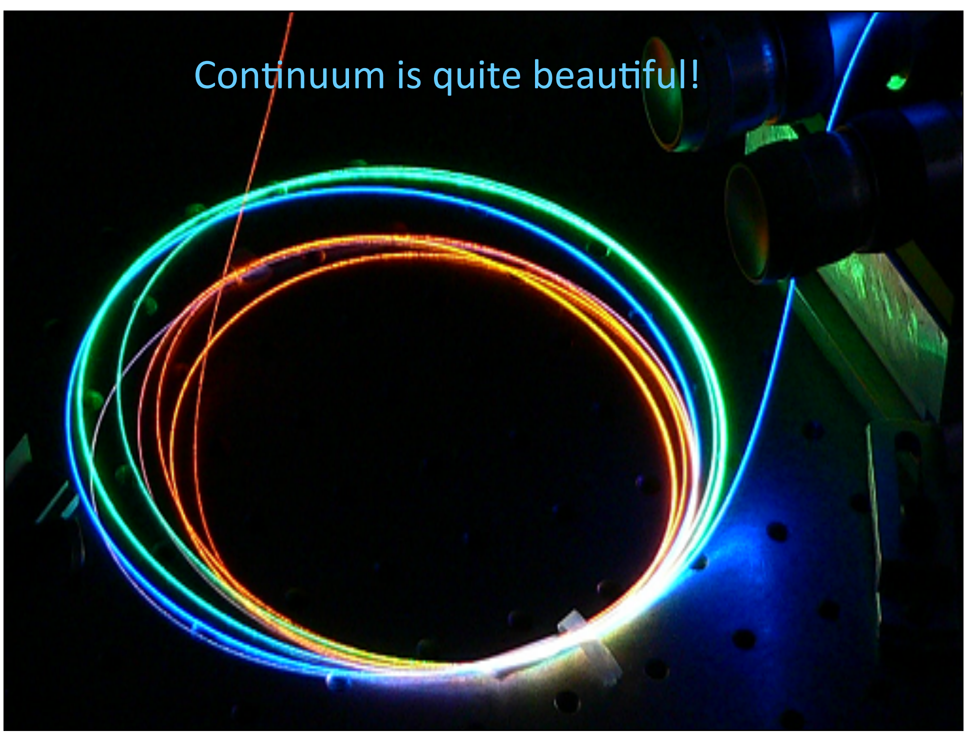


Cross section of the microstructure fiber.

- The spectrum extends from ~400 to ~1500 nm and is relatively flat (when averaged over time).

This continuum was created using *unamplified* Ti:Sapphire pulses.

J.K. Ranka, R.S. Windeler, and A.J. Stentz, Opt. Lett. Vol. 25, pp. 25-27, 2000



Few-Cycle Pulses by External Compression

Sandro De Silvestri, Mauro Nisoli, Giuseppe Sansone, Salvatore Stagira, and Orazio Svelto

F. X. Kärtner (Ed.): Few-Cycle Laser Pulse Generation and Its Applications, Topics Appl. Phys. **95**, 137–178 (2004)

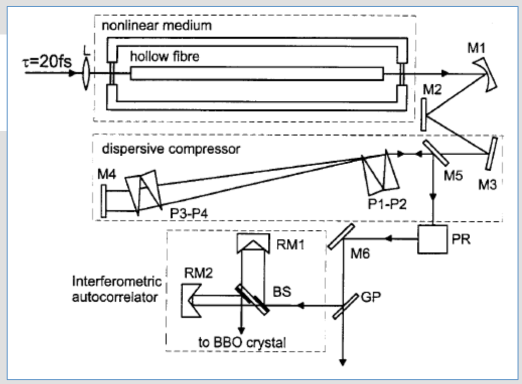
$$\Delta\beta = \frac{(\omega_0/c)\iint \Delta n |F(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y}{\iint |F(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y} .$$

Change in propagation constant is averaged over the mode. SPM applies to whole mode.

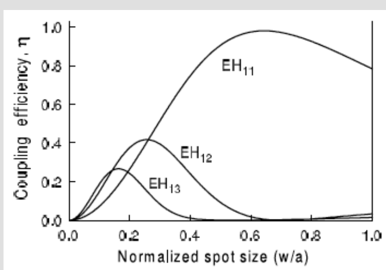
$$\Delta n \propto n_2 \left|F(x,y)\right|^2$$

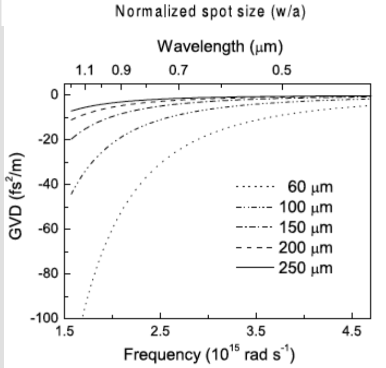
$$\beta(\omega) = \frac{\omega n_{\mathrm{core}}(\omega)}{c}\left[1 - \frac{1}{2}\left(\frac{u_m c}{\omega n_{\mathrm{core}}(\omega) a}\right)^2\right]$$

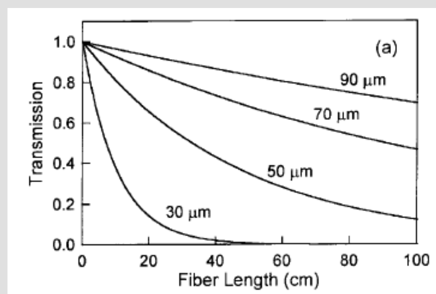
$$+ \frac{\mathrm{i}}{a^3}\left(\frac{u_m c}{\omega n_{\mathrm{core}}(\omega)}\right)^2 \frac{n^2(\omega)+1}{\sqrt{n^2(\omega)-1}}$$

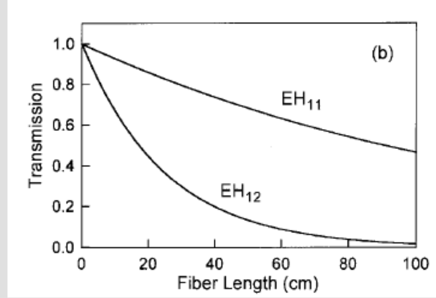


Properties of hollow-core waveguides

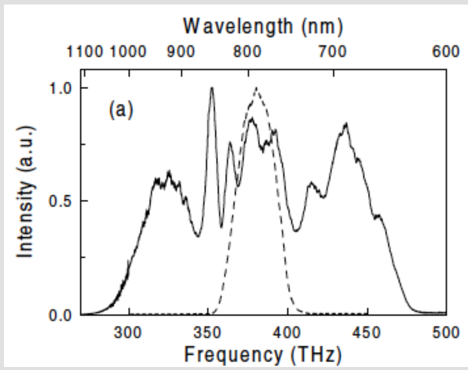


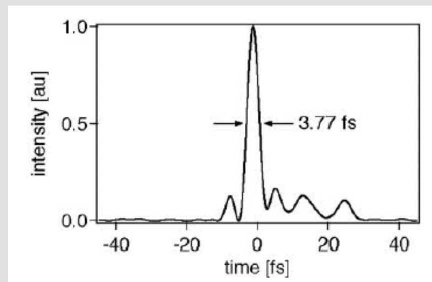






Output spectrum and pulse shape

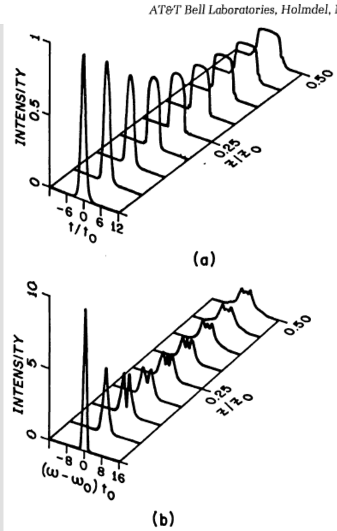


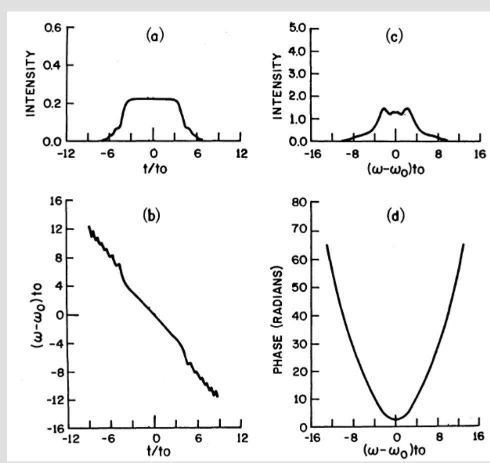


Compression of optical pulses chirped by self-phase modulation in fibers

W. J. Tomlinson,* R. H. Stolen, and C. V. Shank
AT&T Bell Laboratories, Holmdel, New Jersey 07733

Vol. 1, No. 2/April 1984/J. Opt. Soc. Am. B 139





Nonlinear pulse propagation dynamics

Actual dynamics are complicated:

- The nonlinearity of self-phase modulation broadens the spectrum in a way that is sensitive to the pulse shape
- Dispersion reshapes the pulse, changing the SPM

To accurately describe the propagation, we must simultaneously account for dispersion and nonlinearity

Simplest version:
 Nonlinear Schroedinger equation (NLS)

Maxwell's Equations in a Medium

- The induced polarization, **P**, contains the effect of the medium:

$$\boxed{\vec{\nabla}^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}}$$

- Sinusoidal waves of all frequencies are solutions to the wave equation
- The polarization (**P**) can be thought of as the driving term for the solution to this equation, so the polarization determines which frequencies will occur.
- For linear response, **P** will oscillate at the same frequency as the input.

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \chi \mathbf{E}$$

- In nonlinear optics, the induced polarization is more complicated:

$$\mathbf{P}(\mathbf{E}) = \varepsilon_0 \left(\chi^{(1)} \mathbf{E} + \chi^{(2)} \mathbf{E}^2 + \chi^{(3)} \mathbf{E}^3 + \ldots \right)$$

- The nonlinear terms lead to new frequencies and phase modulation.

Linear propagation of quasi-monochromatic fields

- Earlier we had worked with single-frequency fields, for example: $\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x \cos(k_z z - \omega t)$
- Now we want to work with field with a more general temporal shape.
  - Assume linear polarization, plane waves in z-direction
- For now, look at only the linear part of *P* :

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_L}{\partial t^2} \qquad D = \varepsilon_0 E + P_L$$

- Group linear terms together

$$\rightarrow \frac{\partial^2 E}{\partial z^2} - \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 D}{\partial t^2} = 0 \qquad \frac{1}{\varepsilon_0 \mu_0} = c^2$$

Wave equation in frequency space

- Represent all signals in ω space:

$$E(z,t) = \frac{1}{2\pi}\int E(z,\omega) e^{-i\omega t}\, d\omega$$

$$D(z,t) = \frac{1}{2\pi}\int D(z,\omega) e^{-i\omega t}\, d\omega$$

- Now we can connect *D* and *E*: $D(z,\omega) = \varepsilon_0 \varepsilon(\omega) E(z,\omega)$
- Put these expressions into the WE, do time derivatives inside integral: $\frac{\partial^2}{\partial t^2} E(z,t) = \frac{1}{2\pi}\int E(z,\omega)\left(\frac{\partial^2}{\partial t^2} e^{-i\omega t}\right) d\omega$

$$\frac{\partial^2}{\partial z^2} E(z,\omega) + \varepsilon(\omega)\frac{\omega^2}{c^2} E(z,\omega) = 0 \qquad k^2(\omega) = \varepsilon(\omega)\frac{\omega^2}{c^2}$$

- Now work to get back into time domain.

Field with slowly varying envelope

- We went to ω space to be able to easily include dispersion

$$\frac{\partial^2}{\partial z^2}E(z,\omega)+k^2(\omega)E(z,\omega)=0$$

- Represent field in terms of a slowly-varying amplitude

$$E(z,t)=A(z,t)\left(e^{i(k_0 z-\omega_0 t)}+c.c.\right) \qquad A(z,t)=\frac{1}{2\pi}\int A(z,\omega)e^{-i\omega t}\,d\omega$$

  – By shift theorem:

$$E(z,\omega)=A(z,\omega-\omega_0)e^{ik_0 z}$$

- Put this into the wave equation:

$$\frac{\partial^2}{\partial z^2}\left(A(z,\omega-\omega_0)e^{ik_0 z}\right)+k^2(\omega)A(z,\omega-\omega_0)e^{ik_0 z}=\left(\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}-k_0^2+k^2 A\right)e^{ik_0 z}$$

$$\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}+\left(k^2-k_0^2\right)A=0$$

Taylor expansion of dispersion

- Do a Taylor expansion for $k(\omega)$:

$$k(\omega)=k_0+(\omega-\omega_0)k_1+D \qquad D=\sum_{n=2}^{\infty}\frac{1}{n!}(\omega-\omega_0)^n k_n$$

D includes all high-order dispersion

$$k^2(\omega)=k_0{}^2+2k_0k_1(\omega-\omega_0)+k_1{}^2(\omega-\omega_0)^2+2k_0D+2k_1(\omega-\omega_0)D+D^2$$

small

- Insert this expansion into the ω-domain WE:

$$\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}+\left(k(\omega)^2-k_0^2\right)A=0$$

  – Terms in red cancel,

$$\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}+\left(2k_0k_1(\omega-\omega_0)+k_1{}^2(\omega-\omega_0)^2+2k_0D+2k_1(\omega-\omega_0)D\right)A=0$$

Transform back to time domain

$$\frac{\partial^2 A}{\partial z^2}+2ik_0\frac{\partial A}{\partial z}+\left(2k_0k_1\left(\omega-\omega_0\right)+k_1^{\,2}\left(\omega-\omega_0\right)^2+2k_0D+2k_1\left(\omega-\omega_0\right)D\right)A=0$$

- Now inverse FT to go back to time domain
  - Multiply by $e^{-i(\omega-\omega_0)t}$, integrate
  - Note that $FT^{-1}\left\{\omega^n A(\omega)\right\}=\left(i\frac{\partial}{\partial t}\right)^n\tilde{A}(t)$ $\qquad \tilde{D}=\sum_{n=2}^{\infty}\frac{1}{n!}k_n\left(i\frac{\partial}{\partial t}\right)^n$

$$\frac{\partial^2 \tilde{A}}{\partial z^2}+2ik_0\frac{\partial \tilde{A}}{\partial z}+\left(2ik_0k_1\frac{\partial}{\partial t}-k_1^{\,2}\frac{\partial^2}{\partial t^2}+2k_0\tilde{D}+2ik_1\tilde{D}\frac{\partial}{\partial t}\right)\tilde{A}=0$$

- For now, ignore high-order dispersion

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\frac{\partial}{\partial z}+2ik_0k_1\frac{\partial}{\partial t}-k_1^{\,2}\frac{\partial^2}{\partial t^2}\right)\tilde{A}=0$$

  - This can be simplified by changing to a coordinate system moving with the pulse at the group velocity

Moving reference frame

- Change to reference frame moving at the group velocity

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\left(\frac{\partial}{\partial z}+k_1\frac{\partial}{\partial t}\right)-k_1^{\,2}\frac{\partial^2}{\partial t^2}\right)\tilde{A}=0$$

  - Change coordinates:

$$z'=z \qquad \frac{\partial}{\partial z}=\frac{\partial z'}{\partial z}\frac{\partial}{\partial z'}+\frac{\partial \tau}{\partial z}\frac{\partial}{\partial \tau}=\frac{\partial}{\partial z'}-k_1\frac{\partial}{\partial \tau}$$

$$\tau=t-k_1z \qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial \tau} \qquad \frac{\partial^2}{\partial z^2}=\left(\frac{\partial}{\partial z'}-k_1\frac{\partial}{\partial \tau}\right)^2=\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial \tau}+k_1^{\,2}\frac{\partial^2}{\partial \tau^2}$$

$$\left(\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial \tau}+\cancel{k_1^{\,2}\frac{\partial^2}{\partial \tau^2}}+2ik_0\left(\frac{\partial}{\partial z'}\cancel{-k_1\frac{\partial}{\partial \tau}}\cancel{+k_1\frac{\partial}{\partial \tau}}\right)\cancel{-k_1^{\,2}\frac{\partial^2}{\partial \tau^2}}\right)\tilde{A}=0$$

$$\left(\frac{\partial^2}{\partial z'^2}-2k_1\frac{\partial}{\partial z'}\frac{\partial}{\partial \tau}+2ik_0\frac{\partial}{\partial z'}\right)\tilde{A}=0 \qquad \boxed{\rightarrow\left(\frac{\partial^2}{\partial z'^2}+2ik_0\frac{\partial}{\partial z'}\left(1+i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\right)\tilde{A}=0}$$

Simpler equation for envelope.

Slowly-varying envelope approx: SVEA

- So far, we haven't made any approximation about the duration of the pulse (or its bandwidth)
  - Assuming a carrier frequency doesn't itself introduce approximations
- Compare magnitude of components of equation:
  - In general, the envelope $A(z,t)$ will evolve over some length scale L (e.g. b/c of GVD): $\partial/\partial z' \sim 1/L$

$$\left(\frac{\partial^2}{\partial z'^2} + 2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\right)\tilde{A} = 0 \qquad \frac{\partial^2}{\partial z'^2} \sim \frac{1}{L^2} \qquad 2k_0 \frac{\partial}{\partial z'} \sim \frac{4\pi}{\lambda_0 L}$$

  - So if $L \gg \frac{\lambda_0}{4\pi}$ we can ignore second derivative term

SVEA $\frac{\partial^2}{\partial z'^2} \to 0$ $\quad \boxed{2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\tilde{A} = 0}$

  - Dropping this eliminates any counter-propagating solution: no back-reflections included in this approximation.

SVEA again

- We still have an extra time derivative

$$2ik_0 \frac{\partial}{\partial z'}\left(1 + i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\tilde{A} = 0$$

  - Look at ratio:
  - $v_g \sim v_{ph}$ in order of magnitude

$$\frac{k_1}{k_0} = \frac{dk/d\omega|_{\omega_0}}{n\omega_0/c} = \frac{1}{\omega_0}\frac{v_{ph}}{v_g} \approx \frac{1}{\omega_0}$$

- Timescale for change τ_p $\quad \partial/\partial \tau \sim 1/\tau_p$
  - If $\omega_0\tau_p \gg 1$, we can drop the time derivative.

$$\omega_0 \tau_p \approx 2\frac{\omega_0}{\Delta\omega}$$

  - This approximation requires small fractional bandwidth.

$$\to 2ik_0 \frac{\partial}{\partial z'}\tilde{A} = 0$$

  - All this says is that the pulse shape doesn't change, but we assumed there was no high-order dispersion.

Dispersive propagation in the time domain

- Before changing to the moving coordinate system, we had

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\frac{\partial}{\partial z}+2ik_0k_1\frac{\partial}{\partial t}-k_1^{\,2}\frac{\partial^2}{\partial t^2}+2k_0\tilde{D}+2ik_1\tilde{D}\frac{\partial}{\partial t}\right)\tilde{A}=0 \qquad \tilde{D}=\sum_{n=2}^{\infty}\frac{1}{n!}k_n\left(i\frac{\partial}{\partial t}\right)^n$$

  - In moving ref frame, and with SVEA, this is now:

$$\left(2ik_0\frac{\partial}{\partial z'}+2k_0\tilde{D}+2ik_1\tilde{D}\frac{\partial}{\partial \tau}\right)\tilde{A}=0\rightarrow\left(2ik_0\frac{\partial}{\partial z'}+2k_0\tilde{D}\left(1+i\frac{k_1}{k_0}\frac{\partial}{\partial \tau}\right)\right)\tilde{A}=0$$

  - Term in blue is small as in previous slide, so dispersive propagation follows the equation:

$$\left(2ik_0\frac{\partial}{\partial z'}+2k_0\tilde{D}\right)\tilde{A}=0$$

  - For second-order dispersion only,

$$\tilde{D}=\sum_{n=2}^{\infty}\frac{1}{n!}k_n\left(i\frac{\partial}{\partial t}\right)^n\rightarrow\frac{1}{2!}k_2\left(i\frac{\partial}{\partial t}\right)^2=-\frac{1}{2}k_2\frac{\partial^2}{\partial t^2} \qquad \boxed{\frac{\partial\tilde{A}}{\partial z'}=-i\frac{1}{2}k_2\frac{\partial^2\tilde{A}}{\partial t^2}}$$

Nonlinear propagation

- Polarization has a nonlinear component

$$\frac{\partial^2 E}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}=\mu_0\frac{\partial^2 P_L}{\partial t^2}+\mu_0\frac{\partial^2 P_{NL}}{\partial t^2}$$

- Treat $\mu_0\frac{\partial^2 P_{NL}}{\partial t^2}$ as a source term in all previous eqns.

$$\tilde{P}_{NL}(z,t)=3\varepsilon_0\chi^{(3)}\left|\tilde{A}(z,t)\right|^2\tilde{A}(z,t)e^{i(k_0z-\omega_0t)} \qquad n_2I=\frac{3\chi^{(3)}}{2n_0}\left|\tilde{A}\right|^2$$

- Working with the carrier and envelope:

$$\tilde{P}_{NL}(z,t)=\tilde{p}(z,t)e^{i(k_0z-\omega_0t)}$$

$$\frac{\partial\tilde{P}_{NL}}{\partial t}=\left(-i\omega_0\tilde{p}+\frac{\partial\tilde{p}}{\partial t}\right)e^{i(k_0z-\omega_0t)}=-i\omega_0\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)\tilde{p}\,e^{i(k_0z-\omega_0t)}$$

$$\rightarrow\frac{\partial^2\tilde{P}_{NL}}{\partial t^2}=-\omega_0^{\,2}\left(1+\frac{i}{\omega_0}\frac{\partial}{\partial t}\right)^2\tilde{p}\,e^{i(k_0z-\omega_0t)}\approx-\omega_0^{\,2}\tilde{p}\,e^{i(k_0z-\omega_0t)}$$

Drop red term by SVEA

Nonlinear Schrodinger Equation (NLS)

$$\mu_0 \frac{\partial^2 \tilde{P}_{NL}}{\partial t^2} \approx -\mu_0 \omega_0{}^2 \tilde{p}\, e^{i(k_0 z - \omega_0 t)} = -3\mu_0 \varepsilon_0 \omega_0{}^2 \chi^{(3)} \left|\tilde{A}\right|^2 \tilde{A} e^{i(k_0 z - \omega_0 t)}$$

$$= -3\chi^{(3)} \frac{\omega_0{}^2}{c^2} \left|\tilde{A}\right|^2 \tilde{A} e^{i(k_0 z - \omega_0 t)}$$

- Add NL contribution to RHS:

$$\left(2ik_0 \frac{\partial}{\partial z'} + 2k_0 \tilde{D}\right) \tilde{A} = -3\chi^{(3)} \frac{\omega_0{}^2}{c^2} \left|\tilde{A}\right|^2 \tilde{A}$$

$$\left(i \frac{\partial}{\partial z'} + \tilde{D}\right) \tilde{A} = -\frac{\omega_0}{c} n_2 I \tilde{A}$$

- With only 2nd order term in dispersion:

$$\boxed{\frac{\partial \tilde{A}}{\partial z'} = -i\frac{1}{2} k_2 \frac{\partial^2 \tilde{A}}{\partial t^2} + i\frac{\omega_0}{c} n_2 I \tilde{A}}$$

Operator form

$$\partial_z A_0 = \left[\hat{D} + \hat{N}\right] A_0$$

Self-steepening

Driving term for the NL propagation eqn:

$$\frac{1}{\epsilon_0 c^2} \partial_t^2 P = -\frac{\omega_0^2}{\epsilon_0 c^2} \left(1 + \frac{i}{\omega_0} \partial_t\right)^2 \left[p e^{-i\omega_0 t} + c.c.\right]$$

$$= -3\chi^{(3)} \frac{\omega_0^2}{c^2} \left[\frac{1}{A(r,t)} \left(1 + \frac{i}{\omega_0} \partial_t\right)^2 \left(\left|A(r,t)\right|^2 A(r,t)\right)\right] A(r,t) e^{-i\omega_0 t} + c.c.$$

We set up the equation for the term oscillating with exp(- $i\omega_0 t$), giving the RHS:

$$= -3\chi^{(3)} \frac{\omega_0^2}{c^2} \left[\frac{1}{A} \left(1 + \frac{i}{\omega_0} \partial_t\right)^2 \left(\left|A\right|^2 A\right)\right] A$$

For just self-phase modulation (ignoring time derivatives), and no dispersion:

$$2ik_0 \frac{\partial}{\partial z} A = -3\chi^{(3)} \frac{\omega_0^2}{c^2} |A|^2 A$$

Convert to nonlinear index form:

$$\chi^{(3)} = \frac{1}{3} n_2 4 n_0^2 \varepsilon_0 c \qquad I = 2 n_0 \varepsilon_0 c |A|^2$$

$$2ik_0 \frac{\partial}{\partial z} A = -2 n_2 n_0 \frac{\omega_0^2}{c^2} I\, A$$

or

$$\frac{\partial}{\partial z} A = i \frac{\omega_0}{c} n_2 I\, A$$

Now work with the time derivatives on the RHS:

$$\left(1 + \frac{i}{\omega_0} \partial_t\right)^2 = 1 + \frac{2i}{\omega_0} \partial_t - \frac{1}{\omega_0^2} \partial_t^2$$

We'll keep the first derivative.

$$\frac{\partial}{\partial z} A = i \frac{\omega_0}{c} n_2 \left(1 + 2 \frac{i}{\omega_0} \frac{\partial}{\partial t}\right)(I\, A) = i \frac{\omega_0}{c} n_2 \left(I + 2 \frac{1}{A} \frac{i}{\omega_0} \frac{\partial}{\partial t}(I\, A)\right) A$$

The second form is useful for representing the solution in the form:

$$A(z+h) = \exp\left[h\hat{N}\right] A(z)$$

where

$$\hat{N} = i \frac{\omega_0}{c} n_2 \left(I + 2 \frac{1}{A} \frac{i}{\omega_0} \frac{\partial}{\partial t}(I\, A)\right)$$

Note that we can choose A to have units of $\sqrt{I}$ So that $I = |A|^2$

Now let's expand the derivative:

$$\frac{i}{\omega_0}\frac{1}{A}\partial_t\left(|A|^2 A\right) = \frac{i}{\omega_0}\frac{1}{A}\partial_t\left(A^2 A^*\right)$$

$$= \frac{i}{\omega_0}\frac{1}{A}\left(A^2\partial_t A^* + 2|A|^2\,\partial_t A\right)$$

$$= \frac{i}{\omega_0}\left(A\partial_t A^* + 2A^*\partial_t A\right)$$

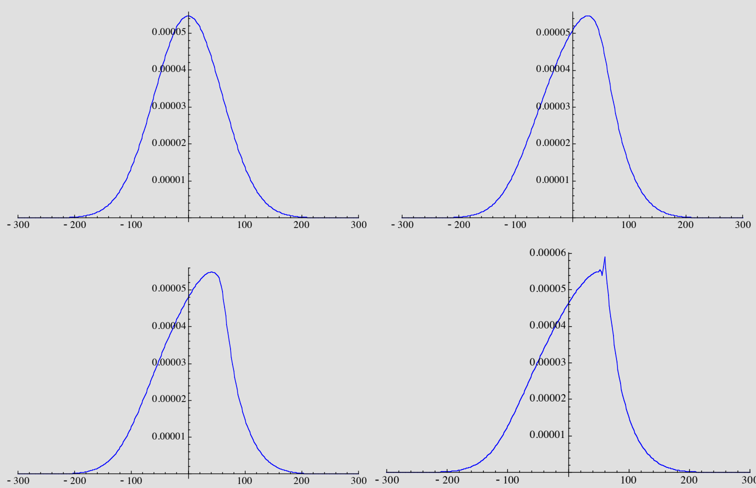
$$A(z+h) = \exp\left[i\frac{\omega_0}{c}n_2\left(|A|^2 + \frac{2i}{\omega_0}\left(A\partial_t A^* + 2A^*\partial_t A\right)\right)h\right]A(z)$$

For the simple case where the pulse envelope is real (no phase term),

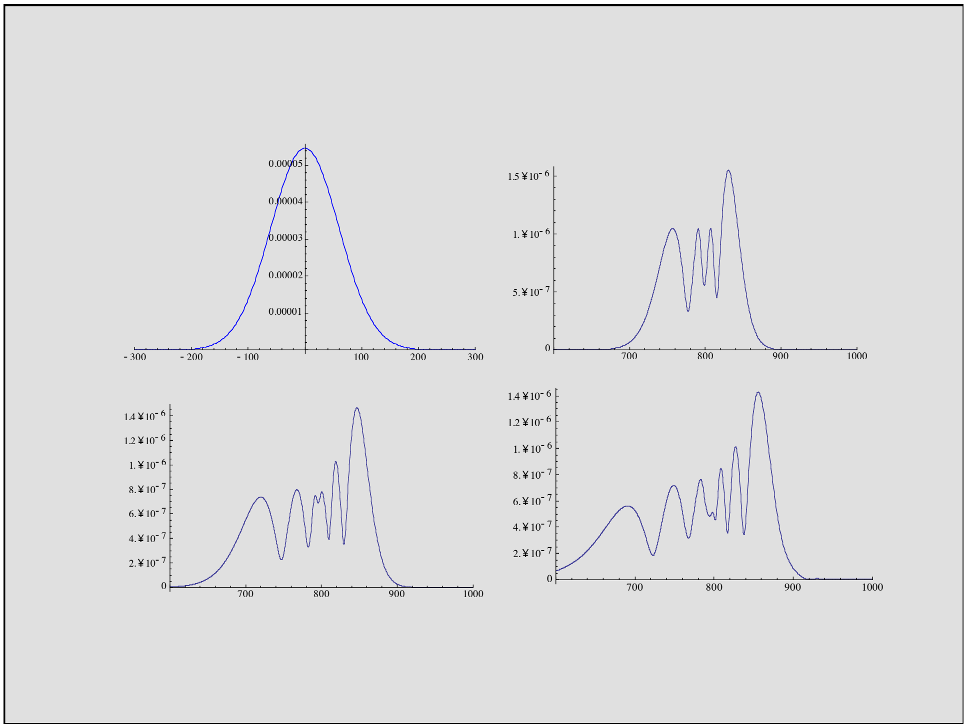
$$A(z+h) = e^{i\frac{\omega_0}{c}n_2|A|^2 h}\exp\left[-\frac{6}{c}(A\partial_t A)h\right]A(z)$$

The first term is the normal SPM term,
the second redistributes power within the pulse

Self-steepening and optical shock formation



Dispersion tends to dissipate the shock.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