MATH 332 - Linear Algebra July 16, 2009

Homework 6, Summer 2009 **Due**: July 20, 2009

Vector Spaces - Subspaces - Bases and Coordiantes - Classical Matrix Spaces - Abstract Vector Spaces

1. (a) Verify that the set of all n-times continuously differentiable functions on $[a, b]$, which satisfies the homogeneous linear ordinary differential equation $L[y] = 0$,

$$V = \left\{ y \in C^{(n)}[a, b] : L[y] = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0(t)y = 0, \text{ where } a_0, \ldots, a_n \in C[a, b] \right\},$$

is a vector space under addition of functions and scalar multiplication.

(b) Prove that if H is the set of all polynomials up to degree n, such that $p(0) = 0$, then H is a subspace of $\mathbb{P}_n$.

(c) Prove that if $H = \{f \in C[a, b] : f(a) = f(b)\}$, then H is a subspace of $C[a, b]$.

2. The standard basis for $\mathbb{R}^2$ are the column vectors, $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{I}_{2\times 2}$. In class we looked at the basis $\mathfrak{B} = \{[1, 1]^{\mathrm{T}}, [-1, 1]^{\mathrm{T}}\}$. This basis is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and does not preserve the notion of length from the standard coordinate system.

(a) Determine a basis for $\mathbb{R}^2$, which is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and preserves the unit length associated with the standard basis.

(b) Show that, for this basis, the change-of-coordinates matrix $\mathbf{P}_{\mathfrak{B}}$ is such that, $\mathbf{P}_{\mathfrak{B}}\mathbf{P}_{\mathfrak{B}}^{\mathrm{T}} = \mathbf{P}_{\mathfrak{B}}^{\mathrm{T}}\mathbf{P}_{\mathfrak{B}} = \mathbf{I}_{2\times 2}$.

(c) Given that $[\mathbf{x}_1]_{\mathfrak{B}} = [\sqrt{2}, \sqrt{2}]^{\mathrm{T}}$ determine $\mathbf{x}_1$ and given that $\mathbf{x}_2 = \left[\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]^{\mathrm{T}}$ determine $[\mathbf{x}_2]_{\mathfrak{B}}$. Calculate the magnitude of both of the vectors previously calculated.

3. Given,

$$\mathbf{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

(a) Is $\mathbf{w}$ in the column space of $\mathbf{A}$? That is, does $\mathbf{w} \in \text{Col } \mathbf{A}$?

(b) Is $\mathbf{w}$ in the null space of $\mathbf{A}$? That is, does $\mathbf{w} \in \text{Nul } \mathbf{A}$?

4. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -1 & 1 \end{bmatrix}.$$

(a) Determine a basis and the dimension of Nul $\mathbf{A}$.

(b) Determine a basis and the dimension of Col $\mathbf{A}$.

(c) Determine a basis and the dimension of Row $\mathbf{A}$.

5. The Hermite polynomials are a sequence of orthogonal polynomials, which arise in probability, combinatorics and physics.[1] The first four polynomials in this sequence are given as,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty, \infty).$$

(a) Show that $\mathfrak{B} = \{1, 2x, -2 + 4x^2, -12x + 8x^3\}$ is a basis for $\mathbb{P}_3$.
Hint: Determine the coordinate vectors of the Hermite polynomials relative to the standard basis.

(b) Let $\mathbf{p}(x) = 7 - 12x - 8x^2 + 12x^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathfrak{B}$.
Hint: Determine $\{c_0, c_1, c_2, c_3\}$ such that $\mathbf{p}(x) = \sum_{i=0}^{3} c_i H_i(\mathrm{x})$.

[1] In physics these polynomials manifest as the spatial solutions to Schrödinger's wave equation under a harmonic potential, which evolves the obability distribution of a quantum mechanical particle near an energy minimum. As it turns out there are infinitely-many Hermite polynomials and nsequently one can show that this particle has infinitely-many allowed quantized energy levels, which are evenly spaced.

3. To show that the set,

$$S = \{ y \in C^{(n)}[a,b] : L[y] = a_0 \frac{d^n y}{dx^n} + \cdots + a_n y = 0, \text{ where } a_0, \cdots, a_n \in C$$

is a vector space.

Proof: We verify all ten axioms.

1. Let $u, v \in S$ then,

$$L[u+v] = a_0(x) \cdot \frac{d^n[u+v]}{dx^n} + a_1(x) \frac{d^{n-1}[u+v]}{dx} + \cdots +$$

$$+ a_n(x)(u+v) = a_0 \frac{d^n u}{dx^n} + a_1(x) \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_n(x) u +$$

$$+ a_0(x) \frac{d^n[v]}{dx^n} + a_1(x) \frac{d^{n-1} v}{dx^n} + \cdots + a_n(x) v =$$

$$= 0 + 0 = 0 \Rightarrow u+v \in S$$

2. Let $u, v \in S$, then since u, v are fxns $u+v = v+u$.
3. Let $w, u, v \in S$, then since u, v are fxns $(u+v)+w = u+(v+w)$

4. Let $u(x) = 0$, $x \in [a,b]$, then $L[u] = L[0] = 0 \Rightarrow u \in S$, and For all $v \in S$, $u+v = 0+v = v$

5. For $u \in S$, let $f = -1 \cdot u$ then

$$(u+f)(x) = (u + -1u)(x) = u(x) - u(x) = 0, \quad x \in [a,b]$$

6. Let $u \in S$, $c \in \mathbb{R}$, then $L[cu] =$

$$= c a_0(x) \frac{d^n u}{dx^n} + c a_1(x) \frac{d^{n-1} u}{dx^{n-1}} + \cdots + c \cdot a_n(x) u =$$

$$= c \cdot \left\{ a_0(x) \frac{d^n u}{dx^n} + a_1(x) \frac{d^{n-1} u}{dx^{n-1}} + \cdots + u \right\} = c \cdot 0 = 0.$$

Thus $cu \in S$.

7. Let $u, v \in S$, $c \in \mathbb{R}$ then by properties of fxns we have,

$$c(u+v)(x) = c(u(x) + v(x)) = cu(x) + cv(x), \quad x \in [a,b]$$

8. Let $c, d \in \mathbb{R}$, $u \in S$, then by properties of fxns we have
$((c+d)u)(x) = cu(x) + du(x)$.

9. Let $c, d \in \mathbb{R}$, $u \in S$ the by properties of fxns we have that
$$(c(du))(x) = c(du(x)) = (cd)u(x), \quad x \in [a,b]$$

10. Let $u \in S$, then
$(1 \cdot u)(x) = 1 \cdot u(x) = u(x), \quad x \in [a,b]$.

4. a. Let H be the set of all polynomials up to degree n, such that for all $p \in H$, $p(0)=0$.

<u>Claim:</u> H is a subspace of $\mathbb{P}_n$.

<u>Proof:</u> Clearly $H \subset \mathbb{P}_n$. Note also that $f(x)=0$, $x \in (-\infty, \infty)$ is in H since $f(0)=0$.

Let $u, v \in H$, $c \in \mathbb{R}$, then

$$(u+v)(0) = u(0) + v(0) = 0+0=0 \Rightarrow u+v \in H.$$

and

$$(cu)(0) = c \cdot u(0) = c \cdot 0 = 0 \Rightarrow cu \in H.$$

Thus H is a subspace of $\mathbb{P}_n$.

b. Let $H = \{ f \in C[a,b] : f(a) = f(b) \}$

<u>Claim:</u> H is a subspace of $C([a,b])$.

<u>Proof:</u> Clearly if $f \in C[a,b]$ then $H \subset C[a,b]$.
Let $g(x)=0$, $x \in [a,b]$. Then $g \in H$ since $g(a)=0=g(b)$.
Thus, H contains the additive identity of $C[a,b]$.

Let $u, v \in H$, $c \in \mathbb{R}$ then

$(u+v)(b) = u(b) + v(b) = u(a) + v(a) = (u+v)(a)$, $u+v \in S$.

and

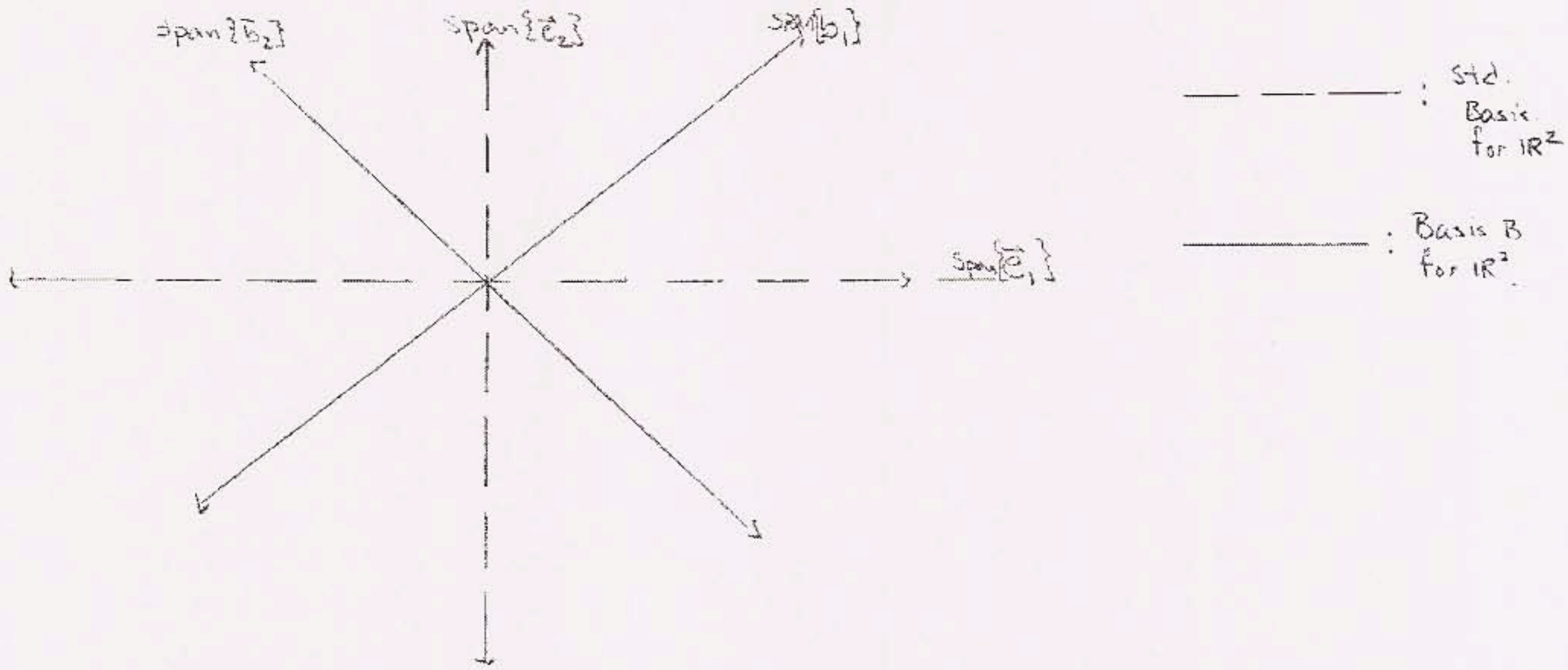
$(cu)(b) = c\,u(b) = c \cdot u(a) = (c \cdot u)(a) \Rightarrow cu \in S$

Thus H is a subspace of $C[a,b]$.

2. To do this we need to use unit vectors. We have 4-vectors which fit the bill.

$\vec{b}_1 = [\,1/\sqrt{2},\ 1/\sqrt{2}\,]^T$
$\vec{b}_2 = [\,-1/\sqrt{2},\ 1/\sqrt{2}\,]^T$
$\vec{b}_3 = [\,+1/\sqrt{2},\ -1/\sqrt{2}\,]^T$
$\vec{b}_4 = [\,-1/\sqrt{2},\ -1/\sqrt{2}\,]^T$

From these 4 we must choose 2 - which are linearly independent. In this case I will choose. $B = \{\vec{b}_1, \vec{b}_2\}$. (What are other possible choi



Why does this basis preserve the unit length?

$\vec{x} = [0, 2]^T$, length of $\vec{x} = \|\vec{x}\| = \sqrt{0^2 + 2^2} = 2$

$$x = P_B[\vec{x}]_B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} [\vec{x}]_B \Rightarrow [\vec{x}_B] = \overbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}^{P_B^{-1}} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$ 4,

and,

the length of $[\vec{x}]_B = \|[\vec{x}]_B\| = \{(\sqrt{2})^2 + (\sqrt{2})^2\}^{1/2} = 2$

So, though the point is described differently in each basis, it's distance to the origin for each basis is still the same.

b. $P_B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $P_B^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. det$(P_B)$

$$P_B P_B^T = \begin{bmatrix} (1/\sqrt{2})^2 + (1/\sqrt{2})^2 & (1/\sqrt{2})^2 - (1/\sqrt{2})^2 \\ (1/\sqrt{2})^2 - (1/\sqrt{2})^2 & (1/\sqrt{2})^2 + (1/\sqrt{2})^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a similar calculation is true for $P_B^T P_B = I$.

If a matrix has this property then it is called unitary. What is the determinate of a unitary matrix?

c. This was found in a. If $[\vec{x}]_B = [\sqrt{2}, \sqrt{2}]^T$ then $\vec{x} = [0, 2]^T$.

d. If $x = [3/\sqrt{2}, 3/\sqrt{2}]^T$ then

$$[\vec{x}]_B = P_B^{-1}\vec{x} = P_B^T\vec{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 3/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

e. This was also done in a. $\|\vec{x}\| = \|[\vec{x}]_B\| = 2$.

5. Let,

$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

a. If $\vec{w} \in \text{Col}\,A$ then $\vec{w} \in \text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Check by Row Reduction.

$$[A \mid \vec{w}] = \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & -2 & -1 & -2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_3 \text{ is free}$$

and x_1, x_2 are uniquely determined in terms of x_3.

Thus

There exists x_1, x_2, x_3 s.t. $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{w}$ and $\vec{w} \in \text{Col}\,A$.

b. $$A\vec{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

thus $\vec{w} \in \text{Nul}\,A$.

3. $A = \begin{bmatrix} 2 & -3 & 6 & 2 & +5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \underset{\substack{R3 = R3 - 2R1 \\ R4 = R4 + R1}}{\overset{R2 = R2 + R1}{\sim}} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \sim$

$\sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & +3 & -1 & +1 \\ 0 & 0 & -3 & +1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \underset{R4 = R4 + 3R3}{\overset{R3 = R3 + R2}{\sim}} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & +3 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim$

$$\sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ \underbrace{0}_{P.C} & 0 & \underbrace{0}_{P.C} & \underbrace{0}_{P.C} & 0 \end{bmatrix} = B$$

a. $B \Rightarrow A\vec{x} = 0$ has the general solution set,

$x_4 = -3x_5$

$x_3 = \frac{(x_4 - x_5)}{3} = \frac{(-3x_5 - x_5)}{3} = \left(\frac{-4}{3}\right)x_5$

$x_1 = \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(\frac{-4}{3}x_5) - 2(-3x_5) - 5x_5) =$

$= \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5$

$x_2 =$ free

$x_5 =$ free. $\quad \Rightarrow \vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \quad x_2, x_5 \in \mathbb{R}$

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Thus the Basis for Nul A is

$$B_{Null} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and $\dim(\text{Nul} A) = \dim B_{Null} = 2$

b. B ⇒ That the basis for the column space of A is the pivot columns $\vec{a}_1, \vec{a}_3, \vec{a}_4$ of A.

$$B_{ColA} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

and

$\dim(\text{Col} A) = \dim B_{ColA} = 3$

c. B ⇒ The Basis for Row A is given as

$$B_{RowA} = \left\{ \begin{array}{l} [2, -3, 6, 2, 5], \\ [0, 0, 3, -1, 1], \\ [0, 0, 0, 1, 3] \end{array} \right\}$$

$\dim(\text{Row} A) = \dim B_{RowA} = 3.$

4. In the standard basis for $\mathbb{P}_3$, $\{1, t, t^2, t^3\}$, the Hermite polynomicals have the following coordinate vectors,

$$(*) \quad H_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

in the standard basis. What would these vectors be in the Basis B?

(*) Implies the transformation matrix,

$$P_B = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

which has 4-pivots and 4 rows. Hence, the columns of P_B are linearly independent and forms a basis for $\mathbb{R}^4$. This implies that (*) and B forms a basis for $\mathbb{P}_3$.

b. Let $p(t) = 7 - 12t - 8t^2 + 12t^3$ we would like to know $\{c_1, c_2, c_3, c_4\}$ s.t.

$$P_B \vec{C} = \begin{bmatrix} 7 \\ -12 \\ -8 \\ 12 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 7 \\ 0 & 2 & 0 & -12 & -12 \\ 0 & 0 & 4 & 0 & -8 \\ 0 & 0 & 0 & 8 & 12 \end{array}\right] \Leftrightarrow$$

$c_4 = 3/2$

$c_3 = -2$

$c_2 = \frac{1}{2}\left(-12 + 12\left(\frac{3}{2}\right)\right) = 3$

$c_1 = 7 + 2(-2) = 3$

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Thus the weights s.t. p is Represented in the B basis is given by $\vec{c} = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 3/2 \end{bmatrix}$

and

$c_1(1) + c_2(2t) + c_3(-2+4t^2) + c_4(-12t+8t^3) =$

$= 3 + 3(2t) + -2(-2+4t^2) + \frac{3}{2}(-12t+8t^3) =$

$= 3+6t + 4 - 8t^2 - 18t + 12t^3 = 7 - 12t - 8t^2 + 12t^3 = p(t)$