# 1. Review of 2nd-Order Constant Linear Homogeneous ODEs

(1.1) $ay'' + by' + cy = 0$  $a, b, c \in \mathbb{R}$

(1.2) $my'' + by' + ky = 0$  $m, b, k \in \mathbb{R}^+$

## 1.1. Quadratic equations and its three cases

To solve (1.2) guess $y = e^{\lambda t}$, $\lambda \in \mathbb{R}$

$\Rightarrow y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$

$$(1.2) \Rightarrow m\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ke^{\lambda t} = 0$$

$$\underbrace{e^{\lambda t}}_{>0 \text{ for all } t}(m\lambda^2 + b\lambda + k) = 0$$

divide by $e^{\lambda t}$ $\Rightarrow m\lambda^2 + b\lambda + k = 0$

Quadratic formula $\Rightarrow$ $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$ (*)

$(m, b, k) = (2, 8, 6)$ : Plugging into (*)

$$\lambda = \frac{-(8) \pm \sqrt{(8)^2 - 4(2)(6)}}{2(2)}$$

Note: $8^2 - 4(2)(6) > 0$

⋮

$$\lambda = -2 \pm 1 = -3, -1$$

From guess $\Rightarrow$ $y = e^{-3t}$, $y = e^{-t}$

Linear Combination of linearly independent solutions

$$\Rightarrow y = C_1 e^{-3t} + C_2 e^{-t}$$

$$y' = -3C_1e^{-3t} - C_2e^{-t}$$

$$y(0)=1 \Rightarrow C_1e^{-3(0)} + C_2e^{-(0)} = 1$$

$$\Rightarrow C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2$$

$$y'(0) = -1 \Rightarrow -3C_1e^{-3(0)} - C_2e^{-(0)} = -1$$

$$\Rightarrow -3C_1 - C_2 = -1$$

$$\Rightarrow -3(1-C_2) - C_2 = -1$$

$$\Rightarrow 2C_2 = 2 \Rightarrow \underline{C_2 = 1}$$

$$\Rightarrow \underline{C_1 = 1-1 = 0}$$

$$\boxed{y = e^{-t}}$$

$(m, b, k) = (1, -4, 13)$ : Plugging into (*)

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

Note: $(-4)^2 - 4(1)(13) < 0$

$$\vdots$$

$$\lambda = 2 \pm 3i$$

From guess $\Rightarrow$ $y = e^{(2+3i)t}$, $y = e^{(2-3i)t}$

$$\left.\begin{aligned} e^{2t}e^{3it} &= e^{2t}(\cos(3t) + i\sin(3t)) \\ e^{2t}e^{-3it} &= e^{2t}(\cos(3t) - i\sin(3t)) \end{aligned}\right\} \text{By Euler's Formula}$$

Linear Combination of Linearly Independent solutions

$$\Rightarrow y = C_1 e^{2t}(\cos(3t) + i\sin(3t)) + C_2 e^{2t}(\cos(3t) - i\sin(3t))$$

$$= e^{2t}\left[\underbrace{(C_1 + C_2)}_{\alpha}\cos(3t) + \underbrace{(C_1 i - C_2 i)}_{\beta}\sin(3t)\right]$$

$$= e^{2t}(\alpha\cos(3t) + \beta\sin(3t))$$

$$y' = 2e^{2t}(\alpha\cos(3t) + \beta\sin(3t)) + e^{2t}(-3\alpha\sin(3t) + 3\beta\cos(3t))$$

$$y(0) = 1 \Rightarrow e^{2(0)}(\alpha\cos(3(0)) + \beta\sin(3(0))) = 1$$

$$\Rightarrow \underline{\alpha = 1}$$

$$y'(0) = -1 \Rightarrow 2e^{2(0)}(\alpha\cos(3(0)) + \beta\sin(3(0))) +$$
$$+ e^{2(0)}(-3\alpha\sin(3(0)) + 3\beta\cos(3(0))) = -1$$

$$\Rightarrow 2(1) + 3\beta = -1 \quad \Rightarrow \underline{\beta = -1}$$

$$\boxed{y = e^{2t}(\cos(3t) - \sin(3t))}$$

$(m, b, k) = (1, 4, 4)$: Plugging into $(*)$

$$\lambda = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(4)}}{2(1)} \qquad \text{Note: } 4^2 - 4(1)(4) = 0$$

$$\lambda = -2$$

From guess $\Rightarrow y = e^{-2t}$

This is only one solution, so to create a second linearly independent solution, we can multiply the first solution by $t$ to get $y = te^{-2t}$

Linear Combination of Linearly Independent Solutions

$$\Rightarrow \quad y = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y' = -2C_1 e^{-2t} - 2C_2 t e^{-2t} + C_2 e^{-2t}$$

$$y(0) = 1 \Rightarrow C_1 e^{-2(0)} + C_2 (0) e^{-2(0)} = 1$$

$$\Rightarrow \underline{C_1 = 1}$$

$$y'(0) = -1 \Rightarrow -2(1)e^{-2(0)} - 2C_2(0)e^{-2(0)} + C_2 e^{-2(0)} = -1$$

$$\Rightarrow -2 + C_2 = -1 \Rightarrow \underline{C_2 = 1}$$

$$\boxed{y = e^{-2t} + te^{-2t}}$$

1.2. The Role of the Coefficient of kinetic friction

$m = 1, \; k = 9$

undamped $\Rightarrow b = 0$

underdamped $\Rightarrow b^2 - 4mk < 0 : b^2 - 4(1)(9) < 0 \Rightarrow b^2 < 36 \Rightarrow b < 6$

critically damped $\Rightarrow b^2 - 4mk = 0 : b^2 - 4(1)(9) = 0 \Rightarrow b^2 = 36 \Rightarrow b = 6$

overdamped $\Rightarrow b^2 - 4mk > 0 : b^2 - 4(1)(9) > 0 \Rightarrow b^2 > 36 \Rightarrow b > 6$

## 1.3. The general case $(my'' + by' + ky = 0)$ (1.2)

As in 1.1., guess $y = e^{\lambda x}$, $\lambda \in \mathbb{R}$

$\Rightarrow y' = \lambda e^{\lambda t}$, $y'' = \lambda^2 e^{\lambda t}$

(1.2) $\Rightarrow m\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + k e^{\lambda t} = 0$

$$\underbrace{e^{\lambda t}}_{>0 \text{ for all } t}(m\lambda^2 + b\lambda + k) = 0$$

divide by $e^{\lambda t}$ $\Rightarrow$ $m\lambda^2 + b\lambda + k = 0$ Note: This is the characteristic polynomial

Quadratic Formula $\Rightarrow$ $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$

Case 1: $b^2 - 4mk > 0$

This gives us two real $\lambda$'s, $\lambda_1$ and $\lambda_2$

$\Rightarrow y_1(t) = e^{\lambda_1 t}$, $y_2(t) = e^{\lambda_2 t}$

Linear Combination of Linearly Independent Solutions

$\Rightarrow \underline{y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$, $c_1, c_2 \in \mathbb{C}$

Now let $\alpha = \frac{-b}{2m}$, $\beta = \frac{\sqrt{b^2-4mk}}{2m}$

$$\Rightarrow \lambda_1 = \alpha + \beta, \quad \lambda_2 = \alpha - \beta$$

Also let $c_1 = \frac{a_1 + a_2}{2}$, $c_2 = \frac{a_1 - a_2}{2}$

$$\Rightarrow y_h(t) = \left(\frac{a_1+a_2}{2}\right)e^{(\alpha+\beta)t} + \left(\frac{a_1-a_2}{2}\right)e^{(\alpha-\beta)t} =$$

$$= \frac{a_1 e^{\alpha t}e^{\beta t} + a_1 e^{\alpha t}e^{-\beta t}}{2} + \frac{a_2 e^{\alpha t}e^{\beta t} - a_2 e^{\alpha t}e^{-\beta t}}{2}$$

$$= a_1 e^{\alpha t}\left(\frac{e^{\beta t} + e^{-\beta t}}{2}\right) + a_2 e^{\alpha t}\left(\frac{e^{\beta t} - e^{-\beta t}}{2}\right)$$

$$\Rightarrow \underline{y_h(t) = a_1 e^{\alpha t}\cosh(\beta t) + a_2 e^{\alpha t}\sinh(\beta t)}$$

Case 2: $b^2 - 4mk < 0$

This gives us two complex $\lambda$'s, $\lambda_1$ and $\lambda_2$,

So $\lambda_1$ and $\lambda_2$ can be written as

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, where $\alpha = \frac{-b}{2m}$, $\beta = \frac{\sqrt{4mk - b^2}}{2m}$

$$\Rightarrow y_1(t) = e^{(\alpha+\beta i)t}, \quad y_2(t) = e^{(\alpha-\beta i)t}$$

Linear Combination of Linearly Independent Solutions

$$\Rightarrow y_h(t) = c_1 e^{(\alpha+\beta i)t} + c_2 e^{(\alpha-\beta i)t}, \quad c_1, c_2 \in \mathbb{C}$$

$$\Rightarrow y_h(t) = c_1 e^{\alpha t} e^{\beta i t} + c_2 e^{\alpha t} e^{-\beta i t} = \quad \text{(By Euler's Formula)}$$

$$= c_1 e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right) + c_2 e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right) =$$

$$= (c_1 + c_2) e^{\alpha t}\cos(\beta t) + (c_1 i - c_2 i) e^{\alpha t}\sin(\beta t)$$

Let $a_1 = c_1 + c_2$, $a_2 = c_1 i - c_2 i$

$$\Rightarrow \underline{y_h(t) = a_1 e^{\alpha t}\cos(\beta t) + a_2 e^{\alpha t}\sin(\beta t)}$$

Case 3: $b^2 - 4mk = 0$

This gives us one real $\lambda = \frac{-b}{2m}$

$$\Rightarrow y_1(t) = e^{\lambda t}, \quad y_2(t) = t e^{\lambda t}$$

Linear Combination of Linearly Independent Solutions

$$\Rightarrow \underline{y_h(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}}, \quad c_1, c_2 \in \mathbb{C}$$

# 2. Power Series, "Trig Functions", and Boundary Conditions

$$(2.1)\quad y'' + \lambda y = 0 \ , \quad \lambda \in \mathbb{R}$$

## 2.1. Intuition

$\lambda > 0$ , $y'' = -\lambda y$ , $\boxed{\sin / \cos}$

- Take two derivatives of sine or cosine and you get a negative constant times the function you started with

$\lambda = 0$ , $y'' = 0$ , $\boxed{ax + b}$

- Take two derivatives of a linear function and you get zero

$\lambda < 0$ , $y'' = -\lambda y = k^2 y$ , $k \in \mathbb{R}$ , $\boxed{\cosh / \sinh \ (\exp)}$

- Take two derivatives of cosh or sinh and you get a positive multiple of the function you started with.

Note: $\cosh = \dfrac{e^x + e^{-x}}{2}$

$\sinh = \dfrac{e^x - e^{-x}}{2}$

## 2.2. Power Series Solution to ODE

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n \cdot n(n-1) x^{n-2}$$

$$y'' + \lambda y = \sum_{n=0}^{\infty} a_n \cdot n(n-1)x^{n-2} + \lambda \sum_{n=0}^{\infty} a_n x^n =$$

$$= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \lambda \sum_{n=0}^{\infty} a_n x^n =$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \lambda \sum_{n=0}^{\infty} a_n x^n =$$

$$= \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + \lambda a_n\right] x^n = 0$$

To make this sum zero for all $x$,

$$a_{n+2}(n+2)(n+1) + \lambda a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{-\lambda a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots$$

Note: This is called the recurrence relation

Case $\lambda = 0$: $a_{n+2} = 0 \Rightarrow a_2 = a_3 = \cdots = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1$$

$$\Rightarrow \underline{y(x) = a_0 + a_1 x}, \quad a_0, a_1 \in \mathbb{R}$$

Case $\lambda \neq 0$: $a_2 = \frac{-\lambda a_0}{2 \cdot 1}$, $a_4 = \frac{-\lambda a_2}{4 \cdot 3} = \frac{\lambda^2 a_0}{4 \cdot 3 \cdot 2 \cdot 1}$

$a_3 = \frac{-\lambda a_1}{3 \cdot 2 \cdot 1}$, $a_5 = \frac{-\lambda a_3}{5 \cdot 4} = \frac{\lambda^2 a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

If we were to continue this process, we would see that odd a's relate to $a_1$ and even a's relate to $a_0$. We can define these even and odd a's as follows:

$$a_{2k} = \frac{(-1)^k \lambda^k}{(2k)!} a_0 \quad , \quad a_{2k+1} = \frac{(-1)^k \lambda^k}{(2k+1)!} a_1$$

From our guess of $y(x) = \sum_{n=0}^{\infty} a_n x^n$, we can split the series into even and odd terms

$$y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} =$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n+1}}{(2n+1)!} \quad , \quad a_0, a_1 \in \mathbb{R}$$

$$\lambda > 0: \quad \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda^{1/2} x)^{2n}}{(2n)!} = \cos(\sqrt{\lambda}\, x)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda^{1/2} x)^{2n+1}}{\lambda^{1/2}(2n+1)!} = \sin(\sqrt{\lambda}\, x)$$

$$\Rightarrow \underline{y(x) = a \cos(\sqrt{\lambda}\, x) + b \sin(\sqrt{\lambda}\, x)} \quad , \quad a, b \in \mathbb{R}$$

Note: $a = a_0$

$b = \frac{a_1}{\lambda^{1/2}}$ , this works because $\lambda^{1/2}$ is just a constant

$$\lambda < 0: \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(\sqrt{-\lambda}\, x)^{2n}}{(2n)!} = \cosh(\sqrt{-\lambda}\, x)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(\sqrt{-\lambda}\, x)^{2n+1}}{\sqrt{-\lambda}\,(2n+1)!} = \frac{\sinh(\sqrt{-\lambda}\, x)}{\sqrt{-\lambda}}$$

Note: $\lambda < 0 \Rightarrow -\lambda > 0$,

also notice that cosh and sinh have series that look just like the series for cos and sine without the alternating term

$$\Rightarrow \underline{y(x) = a\cosh(\sqrt{-\lambda}\, x) + b\sinh(\sqrt{-\lambda}\, x)}, \quad a, b \in \mathbb{R}$$

Note: $a = a_0$

$b = \frac{a_1}{\sqrt{-\lambda}}$, this works because $\sqrt{-\lambda}$ is just a constant

## 2.3. Introduction to Boundary-Value Problem

From 2.2, we know that the solution to $y'' + \lambda y = 0$ can take three forms depending on $\lambda$.

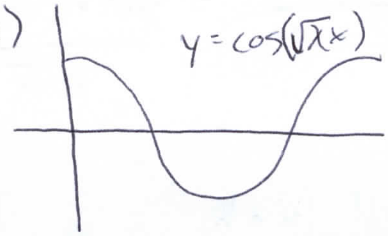
$\lambda > 0$: $y = c_1\cos(\sqrt{\lambda}\, x) + c_2\sin(\sqrt{\lambda}\, x)$

$\lambda < 0$: $y = c_3\cosh(\sqrt{\lambda}\, x) + c_4\sinh(\sqrt{\lambda}\, x)$

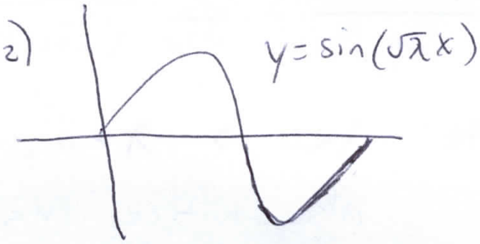
$\lambda = 0$: $y = c_5 x + c_6$

Graphs of the six functions corresponding to the six unknown coefficients:

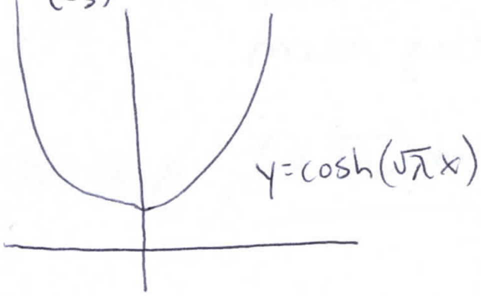
$(c_1)$



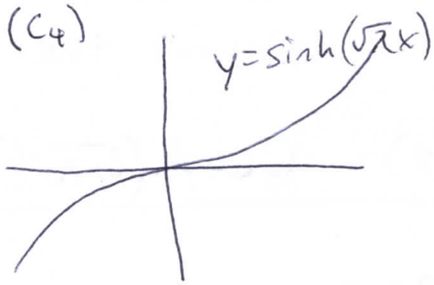

$(c_2)$



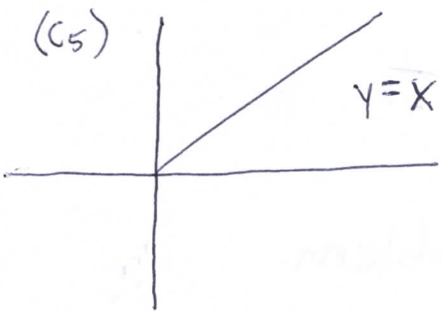

$(c_3)$



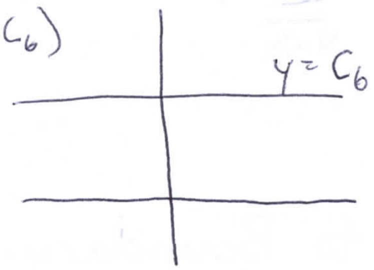

$(c_4)$




$(c_5)$




$(c_6)$




We now have the following Boundary Conditions:

$$l_1 y(0) + l_2 y'(0) = 0$$

$$r_1 y(L) + r_2 y'(L) = 0$$

Case 1: $l_1 = r_1 = 1$, $l_2 = r_2 = 0 \Rightarrow y(0) = y(L) = 0$

This means the function must cross the x-axis at $x = 0$ and $x = L$.

Looking at the graphs, we can see that $y = C_2 \sin(\sqrt{\lambda}\, x)$ is the only nontrivial function that crosses the x-axis at $x=0$ and another point (which we will call $L$), $y = C_6 = 0$ satisfies these B.C. but is trivial.

$y(0) = 0 \Rightarrow C_2 \sin(\sqrt{\lambda}(0)) = 0$

$\Rightarrow 0 = 0$, which tells us nothing

$y(L) = 0 \Rightarrow C_2 \sin(\sqrt{\lambda}(L)) = 0$

$\Rightarrow \sin(\sqrt{\lambda}\, L) = 0$ Note: $C_2 = 0 \Rightarrow y = 0$ is trivial

$\Rightarrow \sqrt{\lambda}\, L = n\pi$, $n \in \mathbb{N}$

$\Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{L}$

$\Rightarrow \lambda = \left(\dfrac{n\pi}{L}\right)^2$

$$\Rightarrow \boxed{y(x) = C_2 \sin\left(\frac{n\pi}{L} x\right)}$$

Case 2: $l_2 = r_2 = 1$, $l_1 = r_1 = 0 \Rightarrow y'(0) = y'(L) = 0$

This means the function must have zero slope at $x=0$ and $x = L$

Looking at the graphs, we can see that $y = C_1\cos(\sqrt{\lambda}x)$ and $y = C_6$ satisfy the B.C.

$y' = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)$ , $0$

$y'(0) = 0 \Rightarrow -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}(0)) = 0$

$\Rightarrow$ $0 = 0$, which tells us nothing

$y'(L) = 0 \Rightarrow -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}(L)) = 0$

$\Rightarrow \sin(\sqrt{\lambda}L) = 0$ Note: $C_1 = 0 \Rightarrow y' = 0$

$\Rightarrow \sqrt{\lambda}L = n\pi$, $n \in \mathbb{N}$ $\Rightarrow y = C_6$

$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{L}$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

$\Rightarrow \boxed{y(x) = C_1\cos\left(\frac{n\pi}{L}x\right), \quad y(x) = C_6}$

Case III: $l_1 = r_2 = 1$, $l_2 = r_1 = 0 \Rightarrow y(0) = y'(L) = 0$

This means that the function must cross the x-axis at $x = 0$ and must have a slope of 0 at some other point which we will call L.

Looking at the graphs, we can see that $y = C_2\sin(\sqrt{\lambda}x)$ is the only non-trivial function to satisfy these conditions. Note: $y = C_6 = 0$ satisfies these conditions, but is a trivial solution.

$y = C_2 \sin(\sqrt{\lambda}\, x)$ , $y' = C_2\sqrt{\lambda} \cos(\sqrt{\lambda}\, x)$

$y(0) = 0 \Rightarrow C_2 \sin(\sqrt{\lambda}\,(0)) = 0$

$\Rightarrow 0 = 0$ , which tells us nothing

$y'(L) = 0 \Rightarrow C_2\sqrt{\lambda} \cos(\sqrt{\lambda}\,(L)) = 0$

$\Rightarrow \cos(\sqrt{\lambda}\,(L)) = 0$ Note: $C_2 = 0 \Rightarrow y = 0$, trivial

$\Rightarrow \sqrt{\lambda}\, L = \frac{n\pi}{2}$ , $n$ is odd

$\Rightarrow \sqrt{\lambda}\, L = \frac{(2n-1)\pi}{2}$ , $n \in \mathbb{N}$ Note: this $n$ is different from the $n$ in the previous step

$\Rightarrow \sqrt{\lambda} = \frac{(2n-1)\pi}{2L}$

$\Rightarrow \lambda = \left[\frac{(2n-1)\pi}{2L}\right]^2$

$\Rightarrow \boxed{y(x) = C_2 \sin\left(\frac{(2n-1)\pi}{2L} x\right)}$

Case IV: $l_2 = r_1 = 1$ , $l_1 = r_2 = 0 \Rightarrow y'(0) = y(L) = 0$

This means that the function must have a slope of zero at $x = 0$ and cross the x-axis at some other point which we will call $L$.

Looking at the graphs, we can see that $y = C_1 \cos(\sqrt{\lambda}\, x)$ is the only non-trivial function to satisfy these conditions. Note: $y = C_6 = 0$ satisfies these conditions, but is a trivial solution.

$y = C_1 \cos(\sqrt{\lambda}\, x)$ , $y' = -C_1\sqrt{\lambda} \sin(\sqrt{\lambda}\, x)$

$y'(0) = 0 \Rightarrow -C_1\sqrt{\lambda} \sin(\sqrt{\lambda}\,(0)) = 0$

$0 = 0$ , which tells us nothing

$y(L) = 0 \Rightarrow C_1 \cos(\sqrt{\lambda}\,(L)) = 0$

$\Rightarrow \cos(\sqrt{\lambda}\,(L)) = 0$ Note: $C_1 = 0 \Rightarrow y = 0$ , trivial

$\Rightarrow \sqrt{\lambda}\, L = \frac{n\pi}{2}$ , $n$ is odd

$\Rightarrow \sqrt{\lambda}\, L = \frac{(2n-1)\pi}{2}$ , $n \in \mathbb{N}$ Note: this $n$ is different from the $n$ in the previous

$\Rightarrow \sqrt{\lambda} = \frac{(2n-1)\pi}{2L}$

$\Rightarrow \lambda = \left[\frac{(2n-1)\pi}{2L}\right]^2$

$\Rightarrow \boxed{y(x) = C_1 \cos\left(\frac{(2n-1)\pi}{2L}\, x\right)}$

# 3. Introduction to Sturm-Liouville Problems

$$L[y] = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y\right)$$

Regular Sturm-Liouville problem if $p$, $p'$, and $q$ are continuous on $[0, L]$ and $p(x) \neq 0$ for all $x \in [0, L]$, singular otherwise.

## 3.1 'Standard' SLP

$p(x) = 1$, $q(x) = 0$, $w(x) = 1$? Find the form of $L[y] = \lambda y$ where

$$L[y] = \frac{1}{(1)}\left(\frac{-d}{dx}\left[(1)\frac{dy}{dx}\right] + (0)y\right) = -\frac{d^2y}{dx^2}$$

$$\Rightarrow L[y] = \lambda y \Rightarrow -\frac{d^2y}{dx^2} = \lambda y \Rightarrow \boxed{y'' = -\lambda y}$$

For what x-values is the problem singular?

$p'(x) = 0$, $p(x) = 1$, $q(x) = 0$ are all continuous everywhere and $p(x) = 1 \neq 0$ for all values of $x$. Therefore, there are no x-values such that this problem is singular.

## 3.2 Bessel's Equation of order $\lambda$.

Find the SL ODE for $p(x) = -q(x) = [-w(x)]^{-1} = x$

$p(x) = x$, $q(x) = -x$, $w(x) = \frac{-1}{x}$

$$L[y] = \frac{1}{\left(\frac{-1}{x}\right)}\left(\underbrace{\frac{-d}{dx}\left[(x)\frac{dy}{dx}\right]} + (-x)y\right)$$

product rule

$$-\left(\frac{d}{dx}(x)\cdot\frac{dy}{dx} + \frac{d}{dx}\left[\frac{dy}{dx}\right](x)\right) = -\frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

$$\Rightarrow L[y] = -x\left(-\frac{dy}{dx} - x\frac{d^2y}{dx^2} - xy\right)$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + x^2 y$$

$$L[y] = \lambda y$$

$$\Rightarrow x^2 y'' + xy + x^2 y = \lambda y$$

$$\Rightarrow \boxed{x^2 y'' + xy + (x^2 - \lambda) y = 0}$$

For what $x$-values is the problem singular?

$p(x) = x$, $p'(x) = 1$, $q(x) = -x$ are continuous everywhere,

and $p(x) = 0$ @ $x = 0$, so this Sturm-Liouville problem is $\boxed{\text{singular at } x = 0.}$

3.3 Legendre's Equation

Find the SL ODE for $p(x) = (1-x)^2$, $q(x) = 0$, $w(x) = 1$.

$$L[y] = \frac{1}{(1)}\left(-\frac{d}{dx}\left[(1-x)^2\frac{dy}{dx}\right] + (0)y\right)$$

$$= -\frac{d}{dx}\left[(1-x)^2\frac{dy}{dx}\right]$$

product rule

$$= -1\left((1-x)^2\frac{d^2y}{dx^2} + 2(1-x)(-1)\frac{dy}{dx}\right)$$

$$= -(1-x)^2 y'' + 2(1-x)y'$$

$$L[y] = \lambda y$$

$$\Rightarrow -(1-x)^2 y'' + 2(1-x)y' = \lambda y$$

$$\Rightarrow -(1-x)^2 y'' + 2(1-x)y' - \lambda y = 0$$

$$\Rightarrow \boxed{(1-x)^2 y'' - 2(1-x)y' + \lambda y = 0}$$

For what $x$-values is the problem singular?

$p(x) = (1-x)^2 \quad p'(x) = 2(1-x)(-1) \quad q(x) = 0$ are all continuous everywhere, and

$$p(x) = (1-x)^2 = 0$$

$$(1-x) = 0$$

$$x = 1$$

$p(x) = 0$ @ $x = 1$, so this Sturm-Liouville problem is $\boxed{\text{singular at } x=1.}$

# 4. Introduction to Bessel's Equation

$$x^2y'' + xy' + x^2y = 0 \quad (*)$$

## 4.1 Power-Series: Step I.

*Note: When $n=0$, the formula evaluates to 0, so we can start the index at 1.

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$

Pull out first term

$$y'(x) = \sum_{n=0}^{\infty} a_n (n)(x)^{n-1} \Rightarrow xy'(x) = \sum_{n=1}^{\infty} a_n(n)(x)^n = a_1 x + \sum_{n=2}^{\infty} a_n(n)(x)^n$$

$$= a_1 x + \sum_{n=0}^{\infty} a_{n+2}(n+2)(x)^{n+2}$$

re-index

$$y''(x) = \sum_{n=0}^{\infty} a_n(n)(n-1)(x)^{n-2} \Rightarrow x^2y''(x) = \sum_{n=2}^{\infty} a_n(n)(n-1)(x)^n$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x)^{n+2}$$

re-index

Plugging these into $(*)$ yields:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x)^{n+2} + a_1 x + \sum_{n=0}^{\infty} a_{n+2}(n+2)(x)^{n+2} +$$

$$+ \underbrace{x^2 \sum_{n=0}^{\infty} a_n x^n}_{= \sum_{n=0}^{\infty} a_n (x)^{n+2}}$$

$$= a_1 x + \sum_{n=0}^{\infty}\left[a_{n+2}(n+2)(x)^{n+2}(n+1) + a_{n+2}(n+2)(x)^{n+2} + a_n(x)^{n+2}\right]$$

$$= a_1 x + \sum_{n=0}^{\infty} (x)^{n+2}\underbrace{\left(a_{n+2}(n+2)((n+1)+1) + a_n\right)}_{a_{n+2}(n+2)^2 + a_n} = 0$$

For this equation to be true for all $x$, all the coefficients must be 0 (i.e. it is a linear combination of linearly independent terms). Therefore:

$$a_1 = 0 \quad \underset{\text{"and"}}{\&} \quad a_{n+2}(n+2)^2 + a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)^2}$$

$$a_3 = \frac{-a_1}{(3)^2} = 0$$

$$a_5 = \frac{-a_3}{(5)^2} = \frac{-(-a_1)}{(5)^2(3)^2} = 0$$

As you can see, all odd subscripts will reduce down to 0, therefore $a_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$

## 4.2 Power Series: Step 2

Show that $a_{2k} = \frac{(-1)^k}{2^{2k}(k!)^2} a_0$.

We know from 4.1 that $a_{n+2} = \frac{-a_n}{(n+2)^2}$. Looking at the even coefficients we get:

$$a_2 = \frac{-a_0}{(0+2)^2} = \frac{-a_0}{2^2} = a_{2\cdot 1}$$

$$a_4 = \frac{-a_2}{(2+2)^2} = \frac{-(-a_0)}{(2+2)^2 \cdot 2^2} = \frac{a_0}{4^2 \cdot 2^2} = \frac{a_0}{(2\cdot 2)^2 \cdot 2^2} = \frac{a_0}{(2^2)^2(2\cdot 1)^2} = a_{2\cdot 2}$$

$$a_6 = \frac{-a_4}{(4+2)^2} = \frac{-(-a_2)}{6^2 \cdot 4^2} = \frac{-(-(-a_0))}{6^2 \cdot 4^2 \cdot 2^2} = \frac{-a_0}{(3\cdot 2)^2 \cdot (2\cdot 2)^2 \cdot (2\cdot 1)^2} = \frac{-a_0}{(2^2)^3 \cdot (3\cdot 2\cdot 1)^2} = a_{2\cdot 3}$$

$$\Rightarrow a_{2k} = \frac{(-1)^k a_0}{(2^2)^k \cdot (k!)^2} = \frac{(-1)^k a_0}{2^{2k}(k!)^2}$$

Setting $a_0 = 1$ we get $a_{2k} = \frac{(-1)^k (1)}{2^{2k} (k!)^2}$

$$J_0(x) = y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \underbrace{\sum_{n=0}^{\infty} a_{2n} x^{2n}}_{\text{even terms}} + \underbrace{\sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}}_{\text{odd terms}}$$

$$= \sum_{n=0}^{\infty} a_{2n} x^{2n} + 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{2^{2n} (n!)^2}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}}$$

**5:** $2^{nd}$ Order Linear ODE Typically, one arrives at the second-order linear ODE,

$$a(x)y'' + b(x)y' + c(x)y = f(x), \tag{1}$$

from Newton's or Kirchoff's law.

**5.1:** Second Linearly Independent Solution Suppose that $a(x) = 1,\ b(x) = 4,\ c(x) = 4,\ f(x) = e^{-2x}$. [1] We know a solution to this problem is $y_1(x) = e^{-2x}$. Using the formula,

$$y_2(x) = k(x)y_1(x),\ k(x) = \int \frac{p(x)}{[y_1(x)]^2}dx,\ p(x) = e^{-\int(b(x)/a(x))dx}, \tag{2}$$

find a second linearly independent solution to the ODE.

We start by noting that, $p(x) = \alpha_2 e^{-4x}$ where $\alpha_2 = e^{-\alpha_1}$, $\alpha_1 \in \mathbb{R}$. Then $k(x) = \int dx = \alpha_2(x + \alpha_3) =$ and formally $y_2(x) = \alpha_2 x e^{-2x} + \alpha_2\alpha_3 e^{-2x}$. However, since $y_h(x) = \beta_1 y_1 + \beta_2 y_2$ we only need the linearly independent portion and $y_2(x) = xe^{-2x}$.

**5.2:** Particular Solution: Part I Using the formula,

$$y_p(x) = y_2 \int \frac{f(x)y_1(x)}{W(x)}dx - y_1 \int \frac{f(x)y_2(x)}{W(x)}dx,\ W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x), \tag{3}$$

find the form for the particular solution. [2]

If notice that $W(x) = e^{-4x}$ then we have the particular solution as,

$$y_p(x) = y_2 \int dx - y_1 \int x dx \tag{4}$$

$$= \frac{x^2 e^{-2x}}{2} \tag{5}$$

**5.3:** Particular Solution: Part II With our newfound trust, we use the previous formula on a problem that couldn't have been analyzed through previous methods. Solve the previous ODE where $a(x) = 1,\ b(x) = 0,\ c(x) = 1,\ f(x) = \sec(x)$, where $x > 0$.

Having solved $y'' + y = 0$ in class we quote the result $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$ and note $W(x) = 1$. Thus,

$$y_p(x) = y_2 \int dx - y_1 \int \tan(x)dx \tag{6}$$

$$= x\sin(x) + \cos(x)\ln|\cos(x)|. \tag{7}$$

[1] This problem is degenerate in the sense that it contains a repeated eigenvalue. Worse, the inhomogeneous term competes with the associated eigenfunction. You can solve this completely using techniques from your previous course work. We will use some formula to justify these techniques.

[2] You might notice that this can be done via the method of undetermined coefficients, which is considerably easier even if you have to multiply your 'guess' by two factors of $x$!