

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points)

(a) True/False : Mark each statement as either true or false.

- i. Suppose that $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, has no solutions. The corresponding homogeneous system, $\mathbf{Ax} = \mathbf{0}$, has only the trivial, $\mathbf{x} = \mathbf{0}$, solution.

False

- ii. If $\mathbf{A} \in \mathbb{R}^{m \times n}$ has a row of zeros then $\mathbf{Ax} = \mathbf{0}$ always has infinitely-many solutions.

False

[See HW #1 p2 matrix A₃]

- iii. It is impossible for a vector to be in both the null-space and column-space of a matrix.

False

[See HW #2 p 1.5]

- iv. If the dimension of the column-space of $\mathbf{A}_{n \times n}$ is n then $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.

True

- v. The system $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A} \in \mathbb{R}^{3 \times 4}$ has only the trivial solution.

False

[See Slide on underdetermined Systems]

(b) Short Response : Provide a short justification of your conclusion.

- i. Suppose $\mathbf{V}_{n \times n}$ is a matrix whose columns form a basis for \mathbb{R}^n . What can be said about the determinant of \mathbf{V} ?

*Columns of V \Rightarrow V ~ I $\Rightarrow \det(V) \neq 0$
 form a basis
 for \mathbb{R}^n*

- ii. Suppose that $\lambda = 0$ is an eigenvalue of \mathbf{A} . What can be said about \mathbf{A}^{-1} ?

$\det(\mathbf{A} - \lambda \mathbf{I}) = \det(\mathbf{A}) = 0 \Rightarrow \mathbf{A}^{-1}$ does not exist.

or

$\mathbf{A}\bar{x} = \lambda\bar{x} = 0 \cdot \bar{x} \Rightarrow \bar{x}$ is a nontrivial sol_n to $\mathbf{A}_{n \times n} \Rightarrow \mathbf{A}^{-1}$ DNE

2. (10 Points) Quickies

(a) Given,

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array} \right]$$

Determine all values of h and k so that the system has:

i. Exactly one solution

$$h \neq 9, k \in \mathbb{R}$$

ii. Infinitely-many solutions

$$h=9, k=6$$

iii. No solutions

$$h=9, k \neq 6$$

(b) Find all eigenvalues of,

$\left[\begin{array}{c} \text{See Hw #3} \\ \text{Prob 1 A.} \end{array} \right]$

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (4-\lambda)(1-\lambda)^2 + 1 \cdot + 2(1-\lambda) = 0 \\ &= (1-\lambda)[(4-\lambda)(1-\lambda) + 2] = \\ &= (1-\lambda)[\lambda^2 - 5\lambda + 6] = (1-\lambda)(2-\lambda)(3-\lambda) = 0 \\ &\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \end{aligned}$$

(c) Find all values of h so that the following vectors are linearly independent.

$\left[\begin{array}{c} \text{See Hw #2} \\ \text{problem 1.2} \end{array} \right]$

$$x = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad y = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & -7 & h+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & -7 & h+3 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h+10 \end{bmatrix} \Rightarrow h \neq -10$$

(d) Find the general solution to the following linear system of equations.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 4 & 5 & 6 & 0 \\ 8 & 9 & 10 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 0 \\ 0 & 9 & 10 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \sim$$

$$\begin{aligned} x_2 + 2x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 8x_1 + 9x_2 + 10x_3 &= 0 \end{aligned}$$

$\left[\begin{array}{c} \text{See Slides} \\ \text{3-planes in} \\ \text{Space} \end{array} \right]$

$$\sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 4 & 5 & 6 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow$$

$$\begin{aligned} x_2 &= -2x_3 \\ 4x_1 - 5x_2 - 6x_3 &= -5(-2x_3) - 6x_3 = 4x_3 \\ \Rightarrow \bar{x} &= \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R} \end{aligned}$$

3. (10 Points) Find a basis for the null-space and column-space of,

[See HW#2
Problem 2]

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{\text{Col}(A)} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\Rightarrow x_4 = -3x_5$$

$$3x_3 = 4x_4 - x_5 = -3x_5 - x_5 = -4x_5 \Rightarrow x_3 = -\frac{4}{3}x_5$$

$$2x_1 = 3x_2 - 6x_3 - 2x_4 - 5x_5 = 3x_2 + 8x_5 + 6x_5 - 5x_5 = 3x_2 + 9x_5 \Rightarrow$$

$$\Rightarrow \vec{x} = \begin{bmatrix} \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ -\frac{4}{3}x_5 \\ -\frac{1}{3}x_5 \\ x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}x_2 + \begin{bmatrix} \frac{9}{2} \\ -\frac{4}{3} \\ -\frac{1}{3} \\ 1 \\ 1 \end{bmatrix}x_5, x_2, x_5 \in \mathbb{R}$$

$$4. (10 Points) \text{ Let } A = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}.$$

$$B_{\text{Null}(A)} = \left\{ \begin{bmatrix} \frac{3}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{9}{2} \\ -\frac{4}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$$

(a) Find the eigenvalues of A.

$$\det(A - \lambda I) = \cancel{2} \cancel{\lambda} \left(\frac{3}{4} - \lambda \right)^2 - \frac{1}{16} = \lambda^2 - \frac{3}{2}\lambda + \frac{9}{16} - \frac{1}{16} = \lambda^2 - \frac{3}{2}\lambda + \frac{8}{16} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$$

$$\lambda = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4(\frac{1}{2})(\frac{1}{2})}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{1}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{1}{2}}{2} = 1, \frac{1}{2}$$

(b) Find the eigenvectors of A.

$$\lambda_1 = 1 \Rightarrow [A - \lambda_1 I | 0] = \left[\begin{array}{cc|c} -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \end{array} \right] \Rightarrow x_1 = x_2 \Rightarrow x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} \Rightarrow \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \end{array} \right] \Rightarrow x^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Calculate $\lim_{n \rightarrow \infty} A^n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} A^n &= \lim_{n \rightarrow \infty} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^n \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

5. (10 Points) Suppose that A has the following eigenvalue, eigenvector pairs.

$$\lambda_1 = 0, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda_3 = -1, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Find the general solution to $A\bar{x} = 0$.

$$1) \quad A\bar{x} = A(c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{e}_3) = c_1\lambda_1\bar{e}_1 + c_2\lambda_2\bar{e}_2 + c_3\lambda_3\bar{e}_3 =$$

$$= c_2\bar{e}_2 + c_3\bar{e}_3 = 0 \Leftrightarrow e_2 \cdot (c_2e_2 + c_3e_3) = e_2 \cdot 0$$

same for e_3 . $\Rightarrow c_2 = 0, c_3 = 0$

$$\Rightarrow c_1 \in \mathbb{R}$$

$$2) \quad A\bar{x} = PDP^{-1}\bar{x} = \bar{0} \Leftrightarrow D\tilde{x} = P^{-1}\bar{0} = \bar{0} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3) \quad A = PDP^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 = \tilde{x}_3 = 0, \tilde{x}_1 \in \mathbb{R} \\ \tilde{x} = P\tilde{x} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tilde{x}_1 \in \mathbb{R} \\ = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} \Rightarrow [A|\bar{0}] = \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 \end{bmatrix} \sim \begin{bmatrix} \tilde{x}_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tilde{x}_1 \in \mathbb{R}$$

$$9) \quad A\bar{x} = \bar{0} \Rightarrow \bar{x}$$

$$A\bar{x} = \lambda\bar{x} = 0 \cdot \bar{x}$$

$$\Rightarrow \bar{x} \text{ is } \bar{e}_1 \text{ up to a constant } \Rightarrow \bar{x} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, c \in \mathbb{R}$$

$$\sim \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \tilde{x}_2 = 0$$

$$\tilde{x}_1 = \tilde{x}_3 \Rightarrow \tilde{x} = \begin{bmatrix} \tilde{x}_3 \\ 0 \\ \tilde{x}_3 \end{bmatrix} = \tilde{x}_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \tilde{x}_3 \in \mathbb{R}$$