MATH 348 - Advanced Engineering Mathematics April 2, 2008
Homework 8, Spring 2008 **Due**: April 11, 2008

PARTIAL DIFFERENTIAL EQUATIONS - THE ONE-DIMENSIONAL WAVE EQUATION

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , \tag{1}$$

$$x \in (0, L), \qquad t \in (0, \infty), \qquad c^2 = \frac{T}{\rho}. \tag{2}$$

Equations (1)-(2) model the time-evolution of the displacement, $u = u(x,t)$, of an elastic medium in one-dimension. The object, of length L, is assumed to have a homogenous lateral tension T, and linear density ρ. That is, $T, \rho \in \mathbb{R}^+$. Define,

$$f(x) = \begin{cases} x, & 0 < x \leq L \\ -x + 2L, & L < x < 2L \end{cases} \tag{3}$$

1. Consider the one-dimensional wave equation, (1)-(2), with the boundary conditions[1],

   $$u_x(0,t) = 0, u(2L,t) = 0, \tag{4}$$

   and initial conditions

   $$u(x,0) = f(x), u_t(x,0) = g(x) \tag{5}$$

   (a) Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (4)-(5). [2]

   (b) Solve for the unknown constants assuming (3) and zero initial velocity for all points on the object.

2. Consider the one-dimensional wave equation, (1)-(2), with the boundary conditions[3],

   $$u_x(0,t) = 0, u_x(2L,t) = 0, \tag{6}$$

   and initial conditions

   $$u(x,0) = f(x), u_t(x,0) = g(x) \tag{7}$$

   (a) Assume that the solution to (1)-(2) is such that $u(x,t) = F(x)G(t)$ and use separation of variables to find the general solution to (1)-(2), which satisfies (6)-(7). [4] [5]

   (b) Let $L = 1$ and solve for the unknown constants assuming (3) and zero initial velocity for all points on the object.

3. Many applications consider traveling wave solutions, $f(x,t) = f(x - ct)$, of the sinusoidal form, $f(x,t) = A\cos(kx - \omega t)$. Assume that $u(x,t) = Ae^{i(kx - \omega t)}$ is a solution to the following wave-like equation:[6]

   $$u_{tt} - u_{xx} + u = 0. \tag{8}$$

   Show that the phase velocity, $c_p = \dfrac{\omega}{k}$, of the traveling wave solutions to (8) is given by $c_p = \pm\sqrt{1 + k^{-2}}$.[7]

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[1] These boundary conditions imply that the object must have zero curvature at the left endpoint and zero displacement at the right endpoint.

[2] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem of hw5 prob1.

[3] These boundary conditions imply that the object must have zero curvature at each endpoint.

[4] It is important to notice that the solution to the spatial portion of the problem is the same as the heat problem of hw5 prob2.

[5] Remember that in this case we have nontrivial solutions for $k_0 = 0$. You should find that $G_0(t) = C_1 + C_2 t$.

[6] Here we choose to work with complex exponential functions since calculation of derivatives is less clumsy than trigonometric functions. Notice that the *real-part* of u is equal to f.

[7] This implies that different waves which solve (8) travel at different velocities.

4. Show that by direct substitution that the function $u(x,t)$ given by,

$$u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy, \tag{9}$$

is a solution to the one-dimensional wave equation where u_0 and v_0 are the initial displacement and velocity of the elastic string, respectively. [8]

5. Consider the non-homogenous one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + F(x,t) \quad , \tag{10}$$

$$x \in (0,L), \qquad t \in (0,\infty), \qquad c^2 = \frac{T}{\rho}. \tag{11}$$

with boundary conditions and initial conditions,

$$u(0,t) = u(L,t) = 0, \tag{12}$$

$$u(x,0) = u_t(x,0) = 0. \tag{13}$$

Letting $F(x,t) = A\sin(\omega t)$ gives the following Fourier Series Representation of the forcing function F ,

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\left(\frac{n\pi x}{L}\right), \tag{14}$$

where

$$f_n(t) = \frac{2A}{n\pi}(1-(-1)^n)\sin(\omega t). \tag{15}$$

(a) Show that substitution of (14)-(15) into (10) gives the ODE,

$$G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n = \frac{2A}{n\pi}(1-(-1)^n)\sin(\omega t). \tag{16}$$

(b) The solution to (16) is given by,

$$G_n(t) = B_n\cos\left(\frac{cn\pi}{L}x\right) + B_n^*\sin\left(\frac{cn\pi}{L}x\right) + G_n^p(t), \tag{17}$$

where $B_n, B_n^* \in \mathbb{R}$ and $G_n^p(t)$ is the particular solution to (16).

i. If $\omega \neq cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

ii. If $\omega = cn\pi/L$ then what would the choice for $G_n^p(t)$ be, assuming you were solving for $G_n^p(t)$ using the method of undetermined coefficients? DO NOT SOLVE FOR THESE COEFFICIENTS

iii. For the latter case what is $\lim_{t\to\infty} u(x,t)$?

iv. What does this limit imply physically?

[8] This is called the D'Alembert solution to the wave equation.