11/10/06

Note Title 11/10/2006

Read 13 Boas

first 4 Sections cover varios examples in cartesian coord.

Laplace' equation

$$\nabla^2 u = 0$$

Poisson's equation

$$\nabla^2 u = f(x,y,z)$$

Diffusion Equation

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

wave equation (scalar)

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

Schrödinger's equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

psi = wavefunction

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plucked string mathem. example on wiki

Comment: why $-\omega^2$ and not $+\omega^2$?

suppose $\dfrac{\ddot{T}}{T} = +\omega^2$

then $\ddot{T} - \omega^2 T = 0$

Solution: $T = T_0 e^{-\omega t}$

whereas

$$\ddot{T} + \omega^2 T = 0$$

Solution: $T = T e^{\pm i\omega t}$

So the physics dictates the sign of the separation const

$-\omega^2 \Rightarrow$ oscillatory solutions

$+\omega^2 \Rightarrow$ expon. decaying

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2 D Case in cartesian

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}$$

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

$\vdots$

$$\frac{\ddot{T}}{T} = -\omega^2$$

$$c^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = -\omega^2$$

The time equation is the same as before.

$$\frac{X''}{X} + \frac{\omega^2}{c^2} = -\frac{Y''}{Y} = \text{const}$$

$$\underbrace{\frac{X''}{X} + \frac{\omega^2}{c^2}}_{\text{depends only on } x} = \underbrace{-\frac{Y''}{Y}}_{\text{depends only on } y}$$

$\frac{\omega^2}{c^2}$ could go on either side

$$\left[\frac{\omega^2}{c^2}\right] = \frac{(1/\text{time})^2}{(\text{dist}/\text{time})^2}$$

$$= \left(\frac{1}{\text{distance}}\right)^2$$

$\frac{1}{\text{distance}}$ = wavenumber

wavenumber (k) is the spatial analog of frequency

$x \Leftrightarrow k$ as $t \Leftrightarrow \omega$

$$\frac{X''}{X} + \frac{\omega^2}{c^2} = -\frac{Y''}{Y} = k_y^2$$

$\Rightarrow$ 2 ODE's

$$\boxed{Y'' + k_y^2 Y = 0}$$

$$X'' + \left(\frac{\omega^2}{c^2} - k_y^2\right) X = 0$$

If we introduce a k_x too then we can let

$$\frac{\omega^2}{c^2} \equiv k^2 = k_x^2 + k_y^2,$$

so that we hav

$$\boxed{\underline{X}'' + k_x^2 \underline{X} = 0}$$

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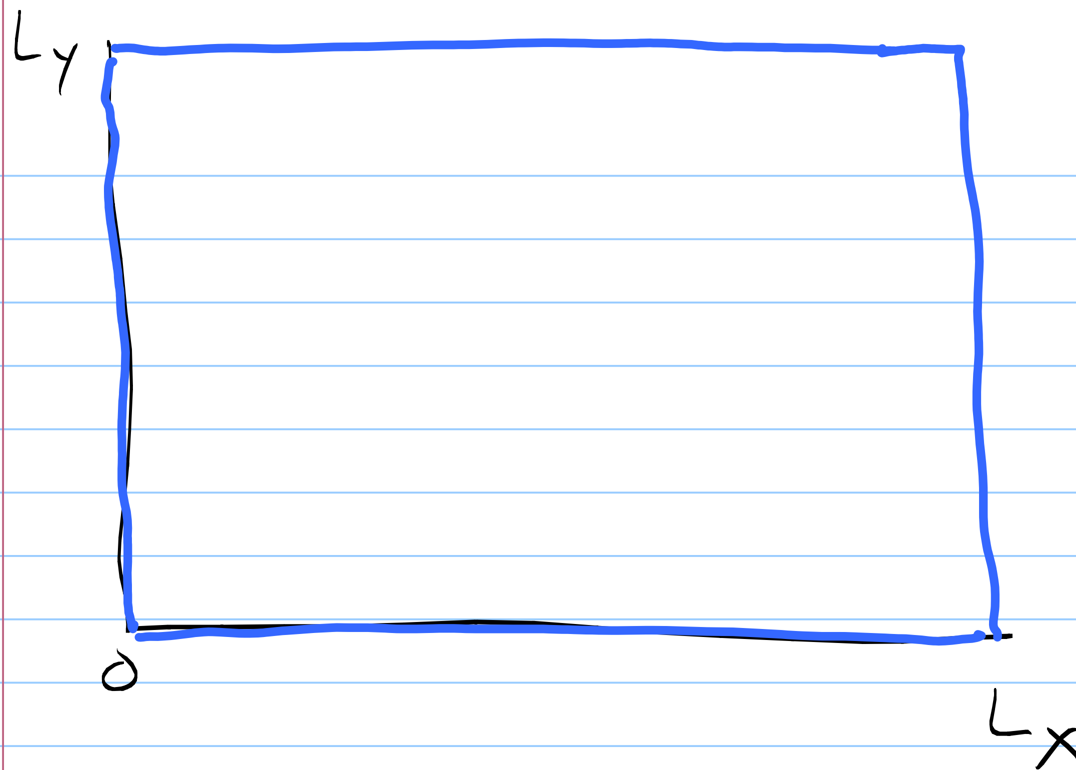
specific case, a rectang. drum (2d analog of a string)



$$\left.\begin{aligned} u(x=0, y, t) &= 0 \\ u(x=L_x, y, t) &= 0 \\ u(x, y=0, t) &= 0 \\ u(x, y=L_y, t) &= 0 \end{aligned}\right\} \text{clamped edges}$$

So, as for the string

$$X(x) = A \sin(k_x x)$$

$$Y(y) = B \sin(k_y y)$$

and $\quad k_x x = \dfrac{n\pi x}{L_x}$

$$k_y y = \frac{m\pi y}{L_y}$$

Since $\dfrac{\omega^2}{c^2} = k_x^2 + k_y^2$, we have

$$\frac{\omega^2}{c^2} = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2$$

or $\boxed{\omega_{n,m}^2 = c^2\pi^2\left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}\right)}$

Spatial part of drum solution $= \mathrm{X}(x)\,\mathrm{Y}(y)$

$$\propto \sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{m\pi y}{L_y}\right)$$

$$\omega_{n,m}^2 = c^2\pi^2\left(\frac{n^2}{L_x^2}+\frac{m^2}{L_y^2}\right)$$

temporal part given by

$$\propto e^{i\omega_{n,m}t}$$