MATH 348 - Advanced Engineering Mathematics October 24, 2009

Homework 7, Summer 2009 **Due**: July 20, 2009

Complex Fourier Series and Introduction to Fourier Transforms

1. Given,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \le \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases} \tag{1}$$

   (a) Sketch a graph f on $[-2L, 2L]$.

   (b) Sketch a graph f^*, the even periodically extension of f, on $[-2L, 2L]$.

   (c) Calculate the Fourier cosine series for the half-range expansion of f.

   Comment: Your graphs from part (a) and part (b) should be different outside of $[0, L]$.

2. In class we derived the complex Fourier series coefficients c_n from the real Fourier series coefficients $a_0,\ a_n,\ b_n$. The coefficients c_n can also be derived using an orthogonality relation similar to the derivations of $a_0,\ a_n,\ b_n$, which is, perhaps, easier.

   (a) First show that $\left\langle e^{inx}, e^{-imx}\right\rangle = 2\pi\delta_{nm}$ where $n, m \in \mathbb{Z}$, where $\langle f, g\rangle = \displaystyle\int_{-\pi}^{\pi} f(x)g(x)dx$.

   (b) Next using the orthogonality relationship defined in (a), find the Fourier coefficients of $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$.

   Hint: For part (b) you should take the inner-product of the complex Fourier series with e^{-imx}. Then, using the orthogonality relation defined in part (a), simplify the doubly-infinite sum to a single term, which gives you your Fourier coefficient c_m. [1]

   Comment: For the complex Fourier series of a $2L$-periodic function you can make a scaling change of variables like we did for the real Fourier series.

3. Let $f(x) = x^2$, $-\pi < x < \pi$, be 2π-periodic.

   (a) Calculate the complex Fourier series representation of f.

   (b) Using the complex Fourier series found in (a), recover the real Fourier series representation of f.

   Hint: For part (b) you will want to follow the example discussed in the class.

[1] Why is $e^{ik\pi} = (-1)^k$ true?

4. Consider the ODE, which is commonly used to model forced simple harmonic oscillation,

$$y'' + 9y = f(t), \tag{2}$$
$$f(t) = |t|, \; -\pi \le t < \pi, \; f(t+2\pi) = f(t). \tag{3}$$

Since the forcing function (3) is a periodic function we can study (2) by expressing $f(t)$ as a Fourier series. [2] [3]

(a) Determine the real-Fourier series representation of $f(t)$.

(b) The solution to the homogenous problem associated with (2) is $y_h(t) = A\cos(3t) + B\sin(3t), \quad A, B \in \mathbb{R}$. Knowing this, if you were to use the method of undetermined coefficients[4] then what would your choice for the particular solution, $y_p(t)$? DO NOT SOLVE FOR THE UNKNOWN CONSTANTS

(c) What is the particular solution associated with the third Fourier mode of the forcing function found in (a)?[5]

(d) What is the long term behavior of the solution to (2) subject to (3)? What if the ODE had the form $y'' + 4y = f(t)$?

5. In differential equations you likely studied the Laplace integral transform, $\int_0^\infty f(t)e^{-st}dt$, which describes how similar the function f is to the transform's *kernel*, i.e. an exponential function. If we consider rotating the kernel $\pi/2$ into the complex plane and integrate over all space then we get a Fourier transform $\int_{-\infty}^{\infty} f(t)e^{-ist}dt$. . If $f(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-i\omega t}d\omega$ then $\hat{f}(\omega)$ describes the complex amplitude of the sinusoid associated with the angular frequency ω. That is, $|\hat{f}(\omega)|^2 = \hat{f}(\omega)\overline{\hat{f}(\omega)}$ gives the square of the amplitude of the sinusoid for ω and thus is an expression of the power carried by that Fourier mode. In preparation of our study of Fourier transforms read the following wikipedia articles,

- `http://en.wikipedia.org/wiki/Fourier_transform` - Opening paragraph, introduction and definition.
- `http://en.wikipedia.org/wiki/Fourier_transform#Cross-correlation_theorem`
- `http://en.wikipedia.org/wiki/Fourier_transform#Uncertainty_principle`

and respond to the following,

(a) How is the Fourier transform related to Fourier series? You should discuss both the periodicity and number of Fourier modes used in the construction of each.

(b) What does cross-correlation measure? What would auto-correlation measure?

(c) What is the uncertainty principle as it relates to Fourier transforms? How much power would be required to send a signal like $\delta(t)$?

[2] The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t. The alternative would have been to construct a solution over each interval $n\pi < t < (n+1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each $t = n\pi$.

[3] It is worth noting that this concepts are used by structural engineers to study the effects of periodic forcing on buildings and bridges.

[4] This is also known as the method of the 'lucky guess' in your differential equations text.

[5] Each term in a Fourier series is called a mode. The first mode is sometimes called the fundamental mode. The rest of the modes, after this fundamental mode, are just referenced by number. The third Fourier mode would be the third term of Fourier summation