E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 7.4, pgs. 296-302
Lecture: Linear Independence and Vector Spaces	Module: 05
Suggested Problem Set: Suggested Problems : {2, 4, 6, 7, 14, 20}	February 2, 2010

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} **ed.**	Section 7.9, pgs. 308-315
Lecture: Abstract V.S. and Inner Product Spaces	Module: 06
Suggested Problem Set: Suggested Problems : { 8, 9, 10, 27, 29 }	February 2, 2010

Quote of Lecture 5	
Springfield Scientist: Let's not listen.	
	The Simpsons: Bye Bye Nerdie (2001)

Again, we study the problem $\mathbf{Ax} = \mathbf{b}$ by asking the question, given $\mathbf{A}$ and $\mathbf{b}$, does a solution exist and is this solution unique. In general the linear system can be solved explicitly using the algorithm of row-reduction. For the case where there exists a unique solution to the system there are many things, which can be said.

1. $\mathbf{A}$ is an invertible matrix
2. $\mathbf{A} \sim \mathbf{I}$
3. $\mathbf{A}$ has n pivot positions
4. $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution
5. $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^n$

The previous statements are equivalent. That is, they are all true or all false. In the case that $\mathbf{Ax} = \mathbf{b}$ does not admit a unique solution then they are all false. How can we characterize the problem in this case?

At this point it makes sense to introduce some of the more abstract concepts in linear algebra. The idea is that if we have unique solutions then we have as many pivots as columns and if we don't have unique solutions[1] then we can expect that pivots$\neq$number-of-columns. Thus it makes sense to determine, given a set of vectors, which are 'important.' We say that the 'important vectors' are the *linearly independent vectors* of that set and more importantly **any** vector in the set can be *constructed* using the linearly independent ones.

Using this concept we then approach the problem in the following way:

1. Given a set of vectors, say the columns of a coefficient matrix, determine which are linearly independent.
2. Construct a *space* of vectors, called the column space of $\mathbf{A}$, defined by the set of all linear combinations of the linearly independent vectors.
3. Ask if the vector $\mathbf{b}$ is in this space. If it is then $\mathbf{Ax} = \mathbf{b}$ has a solution. If not, then no solution exists.

This approach will define some new concepts whose vocabulary is listed below:

- Linear Combination
- Linear Independence
- Rank
- Vector Space

[1] Notice that this would be either no solutions or infinitely many solutions.

- Dimension
- Column Space
- Null Space
- Row Space
- Abstract Vector Space
- Inner Product Space

When this is complete then we will robustly characterize solutions to linear systems in terms of the vocabulary and concepts presented. This will complete our study of chapter 7 with caveat that these concepts underpin the study of all linear problems and will come back again when studying Fourier series and linear PDE.

Goals

- Understand the concepts and vocabulary of vector spaces as they relate solubility of $\mathbf{Ax} = \mathbf{b}$.
- Abstract the concepts of vector spaces to include abstract inner-product spaces.

Objectives

- Calculate the dependence relation given a set of vectors.
- Define the terminology of matrix vector-spaces and how these terms can be realized by determining the linearly independent columns of coefficient matrices.
- Join the concepts of matrix spaces by the rank-nullity theorem and apply this to solubility of $\mathbf{Ax} = \mathbf{b}$.
- Define abstract inner-product spaces and provide examples.