Advanced Engineering Mathematics		**Homework Three**
Eigenproblems : Eigenvalues, Eigenvectors, Diagonalization, Self-Adjoint Operators		
Text: 8.1-8.4	Lecture Notes: 7-8	Lecture Slide: N/A

Quote of Homework Three	
Raoul Duke: Nonsense. We came here to find the American Dream, and now we're right in the vortex you want to quit? You must realize that we've found the Main Nerve.	
	Grisoni and Gilliam : Fear and Loathing in Las Vegas (1998)

1. Eigenvalues and Eigenvectors

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

1.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

2. Diagonalization

2.1. **Eigenbasis and Decoupled Linear Systems.** Find the diagonal matrix $\mathbf{D}_i$ and vector $\tilde{\mathbf{Y}}$ that completely decouples the system of linear differential equations $\dfrac{d\mathbf{Y}_i}{dt} = \mathbf{A}_i\mathbf{Y}_i$ for $i = 3, 4, 5$.

3. Regular Stochastic Matrices

For the *regular stochastic matrix* $\mathbf{A}_4$, define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$.

3.1. **Limits of Time Series.** Show that $\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.

4. Orthogonal Diagonalization and Spectral Decomposition

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ then their inner-product is defined to be $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$. In this case, the 'length' of the vector is $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$.

4.1. **Self-Adjointness.** Show that $\mathbf{A}_5$ is a self-adjoint matrix.

4.2. **Orthogonal Eigenvectors.** Show that the eigenvectors of $\mathbf{A}_5$ are orthogonal with respect to the inner-product defined above.

4.3. **Orthonormal Eigenbasis.** Using the previous definition for length of a vector and the eigenvectors of the self-adjoint matrix, construct an *orthonormal basis* for $\mathbb{C}^2$.

4.4. **Orthogonal Diagonalization.** Show that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, where $\mathbf{U}$ is a matrix containing the normalized eigenvectors of $\mathbf{A}_5$.

4.5. **Spectral Decomposition.** Show that $\mathbf{A}_5 = \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}$.

5. Introduction of Self-Adjoint Operators

A *Sturm-Liouville (SL) Problem* is defined by the linear transformation L,

$$Lu = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right), \tag{1}$$

whose nontrivial solutions must satisfy the boundary conditions,

$$k_1u(a) + k_2u'(b) = 0 \tag{2}$$

$$l_1u(b) + l_2u'(b) = 0. \tag{3}$$

5.1. **Equations in SL Form.** Let $p(x) = 1$, $q(x) = 0$, $w(x) = 1$, $k_1 = l_1 = 0$, $k_2 = l_2 = 1$ and $a = 0$, $b = \pi$. Show that the eigenvalue/eigenfunction pairs to the SLP are defined by $u_n(x) = c_n\cos\left(\sqrt{\lambda_n}x\right)$, $c_n \in \mathbb{R}$, $\lambda_n = n^2$, for $n = 0, 1, 2, 3, \ldots$.

5.2. **Orthogonality of Eigenfunctions.** Using the abstract inner-product defined in homework 2 problem 5.2, $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$, show that the previous eigenfunctions form an orthogonal set. That is, show that $\langle u_n, u_m\rangle = \pi\delta_{nm}$ for $n = 1, 2, 3, \ldots$, and $m = 1, 2, 3, \ldots$.